EMS Series of Lectures in Mathematics
Edited by Andrew Ranicki (University of Edinburgh, U.K.)

EMS Series of Lectures in Mathematics is a book series aimed at students, professional mathematicians and scientists. It publishes polished notes arising from seminars or lecture series in all fields of pure and applied mathematics, including the reissue of classic texts of continuing interest. The individual volumes are intended to give a rapid and accessible introduction into their particular subject, guiding the audience to topics of current research and the more advanced and specialized literature.

Previously published in this series:
Katrin Wehrheim, *Uhlenbeck Compactness*
Torsten Ekedahl, *One Semester of Elliptic Curves*
Sergey V. Matveev, *Lectures on Algebraic Topology*
Joseph C. Várilly, *An Introduction to Noncommutative Geometry*
Reto Müller, *Differential Harnack Inequalities and the Ricci Flow*
Eustasio del Barrio, Paul Deheuvels and Sara van de Geer, *Lectures on Empirical Processes*
Iskander A. Taimanov, *Lectures on Differential Geometry*
Martin J. Mohlenkamp and María Cristina Pereyra, *Wavelets, Their Friends, and What They Can Do for You*
Stanley E. Payne and Joseph A. Thas, *Finite Generalized Quadrangles*
Masoud Khalkhali, *Basic Noncommutative Geometry*
Helge Holden, Kenneth H. Karlsen, Knut-Andreas Lie and Nils Henrik Risebro, *Splitting Methods for Partial Differential Equations with Rough Solutions*
Koichiro Harada, *"Moonshine" of Finite Groups*
Yurii A. Neretin, *Lectures on Gaussian Integral Operators and Classical Groups*
Damien Calaque and Carlo A. Rossi, *Lectures on Duflo Isomorphisms in Lie Algebra and Complex Geometry*

Claudio Carmeli
Lauren Caston
Rita Fioresi

Mathematical Foundations of Supersymmetry

European Mathematical Society

Authors:

Claudio Carmeli
Dipartimento di Fisica
Università di Genova
Via Dodecaneso, 33
16146 Genova
Italy

E-mail: claudio.carmeli@gmail.com

Lauren Caston
RAND Corporation
1776 Main Street
Santa Monica, CA 90407-2138
USA

E-mail: lcaston@rand.org

Rita Fioresi
Dipartimento di Matematica
Università di Bologna
Piazza Porta San Donato 5
40126 Bologna
Italy

E-mail: fioresi@dm.unibo.it

2010 Mathematics Subject Classification: 58-02; 58A50, 58C50, 14M30, 17A70, 32C11, 81Q60

Key words: supermanifolds, supersymmetry, superschemes, Lie superalgebras

ISBN 978-3-03719-097-5

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

Contact address:
European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland
Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the authors' TEX files: I. Zimmermann, Freiburg
Printing and binding: Druckhaus Thomas Müntzer GmbH, Bad Langensalza, Germany
∞ Printed on acid free paper

9 8 7 6 5 4 3 2 1

Preface

Supersymmetry was discovered by physicists in the 1970s. The mathematical treatment of it began much later and grew out of the works notably of Berezin, Kostant, Leites, Manin, Bernstein, Freed, Deligne, Morgan, Varadarajan and others. These works are all in what one may call the differential category and contain many additional references to the subject.

This monograph has grown out of the desire to present a moderately brief and focussed account of the mathematical foundations of supersymmetry both in the differential and algebraic categories. Our view is that supergeometry and super Lie theory are beautiful areas and deserve much attention.

Our intention was not to write an encyclopedic treatment of supersymmetry but to supply only the foundational material that will allow the reader to penetrate the more advanced papers in the wide literature on this subject. For this reason we do not treat the differential and symplectic supergeometry and we are unable to give a comprehensive treatment of the representation theory of Lie supergroups and Lie superalgebras, which can be found in more advanced papers by Kac, Serganova, Penkov, Duflo, Cassinelli et al. and so on.

Our work is primarily directed to second or third year graduate students who have taken a one year graduate course in algebra and a beginning course in Lie groups and Lie algebras. We have provided a discussion without proofs of the classical theory, which will serve as a departure point for our supergeometric treatment. Our book can very well be used as a one-semester course or a participating seminar on supersymmetry, directed to second and third year graduate students.

The language used in this monograph is that of the functor of points. Since this language is not always familiar even to second-year graduate students we have attempted to explain it even at the level of classical geometry. Apart from being the most natural medium for understanding supergeometry, it is also, remarkably enough, the language closest to the physicists' method of working with supersymmetry.

We wish to thank professor V. S. Varadarajan for introducing us to this beautiful part of mathematics. He has truly inspired us through his insight and deep understanding of the subject. We also wish to thank Dr. L. Balduzzi, Prof. G. Cassinelli, Prof. A. Cattaneo, Prof. M. Duflo, Prof. F. Gavarini, Prof. A. Kresch, Prof. M. A. Lledo, Prof. L. Migliorini, Prof. I. M. Musson, Prof. V. Ovsienko, Dr. E. Petracci, Prof. A. Vistoli and Prof. A. Zubkov for helpful remarks. We also want to thank the UCLA Department of Mathematics, the Dipartimento di Matematica, Università di Bologna, and the Dipartimento di Fisica, Università di Genova, for support and hospitality during the realization of this work.

Introduction

Supersymmetry (SUSY) is the machinery mathematicians and physicists have developed to treat two types of elementary particles, *bosons* and *fermions*, on the same footing. Supergeometry is the geometric basis for supersymmetry; it was first discovered and studied by physicists, Wess and Zumino [80], Salam and Strathdee [65] (among others), in the early 1970s. Today supergeometry plays an important role in high energy physics. The objects in super geometry generalize the concept of smooth manifolds and algebraic schemes to include anticommuting coordinates. As a result, we employ the techniques from algebraic geometry to study such objects, namely A. Grothendieck's theory of schemes.

Fermions include all of the material world; they are the building blocks of atoms. Fermions do not like each other. This is in essence the Pauli exclusion principle which states that two electrons cannot occupy the same quantum mechanical state at the same time. Bosons, on the other hand, can occupy the same state at the same time.

Instead of looking at equations that simply describe either bosons or fermions separately, supersymmetry seeks out a description of both simultaneously. Transitions between fermions and bosons require that we allow transformations between the commuting and anticommuting coordinates. Such transitions are called supersymmetries.

In classical Minkowski space, physicists classify elementary particles by their mass and spin. Einstein's special theory of relativity requires that physical theories must be invariant under the Poincaré group. Since observable operators (e.g. Hamiltonians) must commute with this action, the classification corresponds to finding unitary representations of the Poincaré group. In the SUSY world, this means that mathematicians are interested in unitary representations of the super Poincaré group. A "super" representation gives a "multiplet" of ordinary particles which include both fermions and bosons.

Up to this point, there have been no colliders that can produce the energy required to physically expose supersymmetry. However, the Large Hadron Collider (LHC) in CERN (Geneva, Switzerland) became operational in 2007. Physicists are planning proton–proton and proton–antiproton collisions which will produce energies high enough where it is believed supersymmetry can be seen. Such a discovery will solidify supersymmetry as the most viable path to a unified theory of all known forces. Even before the boson–fermion symmetry which SUSY presupposes is proved to be physical fact, the mathematics behind the theory is quite remarkable. The concept that space is an object built out of local pieces with specific local descriptions has evolved through many centuries of mathematical thought. Euclidean and non-Euclidean geometry, Riemann surfaces, differentiable manifolds, complex manifolds, algebraic varieties, and so on represent various stages of this concept. In Alexander Grothendieck's theory of schemes, we find a single structure that encompasses all previous ideas of space. However, the fact that conventional descriptions of space will fail at very small distances (Planck length) has been the driving force behind the discoveries of unconventional models of space that are rich enough to portray the quantum fluctuations of space at these unimaginably small distances. Supergeometry is perhaps the most highly developed of these theories; it provides a surprising application and continuation of the Grothendieck theory and opens up large vistas. One should not think of it as a mere generalization of classical geometry, but as a deep continuation of the idea of space and its geometric structure.

Out of the first supergeometric objects constructed by the pioneering physicists came mathematical models of superanalysis and supermanifolds independently by F.A. Berezin [10], B. Kostant [49], D.A. Leites [53], and De Witt [25]. The idea to treat a supermanifold as a ringed space with a sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded algebras was introduced in these early works. Later, Bernstein [22] and Leites [53] used techniques from algebraic geometry to deepen the study of supersymmetry. In particular, Bernstein and Leites accented the functor of points approach from Grothendieck's theory of schemes. Interest in SUSY has grown in the past decade, and most recently works by V. S. Varadarajan [76] and others have continued exploration of this beautiful area of physics and mathematics and have inspired this work. Given the interest and the number of people who have contributed greatly to this field from various perspectives, it is impossible to give a fair and accurate account of all the works related to ours. We have nevertheless made an attempt and have provided bibliographical references at the end of each chapter, pointing out the main papers that have inspired our work. We apologize for any involuntary omissions.

In our exposition of mathematical SUSY, we use the language of T-points to build supermanifolds up from their foundations in $\mathbb{Z}/2\mathbb{Z}$-graded linear algebra (superalgebra). The following is a brief description of our work.

In Chapter 1 we begin by studying $\mathbb{Z}/2\mathbb{Z}$-graded linear objects. We define super vector spaces and superalgebras, then generalize some classical results and ideas from linear algebra to the super setting. For example, we define a super Lie algebra, discuss supermatrices, and formulate the super trace and determinant (the Berezinian). We also discuss the Poincaré–Birkhoff–Witt theorem in full detail.

In Chapter 2 we provide a brief account of classical sheaf theory with a section dedicated to schemes. This is meant to be an introductory chapter on this subject and the advanced reader may very well skip it.

In Chapter 3 we introduce the most basic geometric structure: a superspace. We present some general properties of superspaces which lead into two key examples of superspaces, supermanifolds and superschemes. Here we also introduce the notion of T-points which allows us to treat our geometric objects as functors; it is a fundamental tool to gain geometric intuition in supergeometry.

Chapters 4–9 lay down the full foundations of C^∞-supermanifolds over $\mathbb{R}$. In Chapter 4, we give a complete proof of foundational results like the chart theorem and the correspondence between morphisms of supermanifolds and morphisms of the superalgebras of their global sections. In Chapter 5 we discuss the local structure of morphisms proving the analog of the inverse function, submersion and immersion theorems. In Chapter 6 we prove the local and global Frobenius theorem on supermanifolds. In Chapters 7 and 8 we give special attention to super Lie groups and their associated Lie algebras, as well as look at how group actions translate infinitesimally. We then use infinitesimal actions and their characterizations to build the super Lie subgroup, subalgebra correspondence. Finally in Chapter 9 we discuss quotients of Lie supergroups.

Chapters 10, 11 expand upon the notion of a superscheme which we introduce in Chapter 3. We immediately adopt the language of T-points and give criteria for representability: in supersymmetry it is often most convenient to describe an object functorially, and then show that it is representable. We explicitly construct the Grassmannian functorially, then use the representability criterion to show that it is a superscheme. Chapter 10 concludes with an examination of the infinitesimal theory of superschemes.

We continue this exploration in Chapter 11 from the point of view of algebraic supergroups and their Lie algebras. We discuss the linear representations of affine algebraic supergroups; in particular we show that all affine supergroups are realized as subgroups of the general linear supergroup.

We have made an effort to make this work self-contained and suggest that the reader begins with Chapters 1–3, but Chapters 4–9 and Chapters 10–11 are somewhat disjoint and may be read independently.

Contents

1

$\mathbb{Z}/2\mathbb{Z}$-graded linear algebra

The theory of manifolds and algebraic geometry are ultimately based on linear algebra. Similarly the theory of supermanifolds needs super linear algebra, which is linear algebra in which vector spaces are replaced by vector spaces with a $\mathbb{Z}/2\mathbb{Z}$-grading, namely, super *vector spaces*. The basic idea is to develop the theory along the same lines as the usual theory, adding modifications whenever necessary. We therefore first build the foundations of linear algebra in the super context. This is an important starting point as we later build super geometric objects from sheaves of super linear spaces.

Let us fix a ground field k, $\mathrm{char}(k) \neq 2, 3$.

1.1 Super vector spaces and superalgebras

Definition 1.1.1. A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$V = V_0 \oplus V_1$$

where elements of V_0 are called "even" and elements of V_1 are called "odd".

Definition 1.1.2. The *parity* of $v \in V$, denoted by $p(v)$ or $|v|$, is defined only on non-zero *homogeneous* elements, that is elements of either V_0 or V_1:

$$p(v) = |v| = \begin{cases} 0 \text{ if } v \in V_0, \\ 1 \text{ if } v \in V_1. \end{cases}$$

Since any element may be expressed as the sum of homogeneous elements, it suffices to consider only homogeneous elements in the statement of definitions, theorems, and proofs.

Definition 1.1.3. The *superdimension* of a super vector space V is the pair (p, q) where $\dim(V_0) = p$ and $\dim(V_1) = q$ as ordinary vector spaces. We simply write $\dim(V) = p|q$.

From now on we will simply refer to the superdimension as the dimension when it is clear that we are working with super vector spaces. If $\dim(V) = p|q$, then we can find a basis $\{e_1, \ldots, e_p\}$ of V_0 and a basis $\{\epsilon_1, \ldots, \epsilon_q\}$ of V_1 so that V is canonically isomorphic to the free k-module generated by the $\{e_1, \ldots, e_p, \epsilon_1, \ldots, \epsilon_q\}$. We denote this k-module by $k^{p|q}$ and we will call $(e_1, \ldots, e_p, \epsilon_1, \ldots, \epsilon_q)$ the *canonical basis* of $k^{p|q}$. The (e_i) form a basis of $k^p = k_0^{p|q}$ and the (ϵ_j) form a basis for $k^q = k_1^{p|q}$.

Definition 1.1.4. A *morphism* from a super vector space V to a super vector space W is a linear map from V to W preserving the $\mathbb{Z}/2\mathbb{Z}$-grading. Let $\mathrm{Hom}(V, W)$ denote the vector space of morphisms $V \to W$.

Thus we have formed the category[1] of super vector spaces that we denote by (smod). It is important to note that the category of super vector spaces also admits an "inner Hom", which we denote by $\underline{\mathrm{Hom}}(V, W)$; for super vector spaces V, W, $\underline{\mathrm{Hom}}(V, W)$ consists of *all* linear maps from V to W; it is made into a super vector space itself by the following definitions:

$$\underline{\mathrm{Hom}}(V, W)_0 = \{T : V \to W \mid T \text{ preserves parity}\} \quad (= \mathrm{Hom}(V, W));$$

$$\underline{\mathrm{Hom}}(V, W)_1 = \{T : V \to W \mid T \text{ reverses parity}\}.$$

If $V = k^{m|n}$, $W = k^{p|q}$ we have, in the canonical basis (e_i, ϵ_j):

$$\underline{\mathrm{Hom}}(V, W)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \quad \text{and} \quad \underline{\mathrm{Hom}}(V, W)_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where A, B, C, D are respectively $(p \times m)$, $(p \times n)$, $(q \times m)$, $(q \times n)$-matrices with entries in k.

In the category of super vector spaces we have the *parity reversing functor* $\Pi(V \to \Pi V)$ defined by

$$(\Pi V)_0 = V_1, \quad (\Pi V)_1 = V_0.$$

The category of super vector spaces admits tensor products: for super vector spaces V, W, $V \otimes W$ is given the $\mathbb{Z}/2\mathbb{Z}$-grading as follows:

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1),$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

The assignment $V, W \mapsto V \otimes W$ is additive and exact in each variable as in the ordinary vector space category. The object k functions as a unit element with respect to tensor multiplication $\otimes$; and tensor multiplication is associative, i.e., the two products $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ are naturally isomorphic. Moreover, $V \otimes W \cong W \otimes V$ by the *commutativity map*

$$c_{V,W} : V \otimes W \to W \otimes V$$

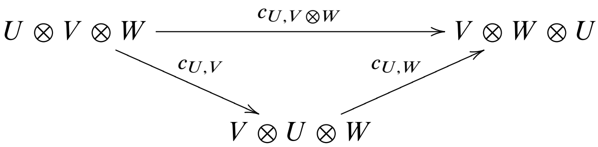
where $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.

The significance of this definition is as follows. If we are working with the category of vector spaces, the commutativity isomorphism takes $v \otimes w$ to $w \otimes v$. In super linear algebra we have to add the sign factor in front. This is a special case of the general

[1] We refer the reader not accustomed to category language to Appendix B.1.

principle called the "sign rule" that one finds in some physics and mathematics literature. The principle says that in making definitions and proving theorems, the transition from the usual theory to the super theory is often made by just simply following this principle, which introduces a sign factor whenever one reverses the order of two odd elements. The functoriality underlying the constructions makes sure that the definitions are all consistent.

The commutativity isomorphism satisfies the so-called *hexagon diagram*:



where, if we had not suppressed the arrows of the associativity morphisms, the diagram would have the shape of a hexagon.

The definition of the commutativity isomorphism, also informally referred to as the sign rule, has the following very important consequence. If $V_1, \dots, V_n$ are super vector spaces and σ and τ are two permutations of n elements, no matter how we compose associativity and commutativity morphisms, we always obtain the same isomorphism from $V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}$ to $V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)}$ namely:

$$V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} \to V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)},$$

$$v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mapsto (-1)^N v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}$$

where N is the number of pairs of indices i, j such that v_i and v_j are odd and $\sigma^{-1}(i) < \sigma^{-1}(j)$ with $\tau^{-1}(i) > \tau^{-1}(j)$.

The *dual*, V^*, of V is defined as

$$V^* := \underline{\mathrm{Hom}}(V, k).$$

Notice that, if V is even, that is $V = V_0$, we have V^* is the ordinary dual of V, consisting of all even morphisms $V \to k$. If V is odd, that is $V = V_1$, then V^* is also an odd vector space and consists of all odd morphisms $V^1 \to k$. This is because any morphism from V_1 to $k = k^{1|0}$ is necessarily odd since it sends odd vectors into even ones.

The category of super vector spaces thus becomes what is known as a *tensor category with inner Hom and dual*. We start by recalling the universal property of the tensor product.

Proposition 1.1.5. *Let V and W be two super vector spaces and f a bilinear map of $V \times W$ into a third super vector space Z. Then there exists a unique morphism $g: V \otimes W \to Z$ such that*

$$g(v \otimes w) = f(v, w) \quad (v \in V,\ w \in W).$$

Proof. See [51], Ch. XVI. □

Remark 1.1.6. The object $V^{\otimes n} = V \otimes \cdots \otimes V$ (n times) for a super vector space V is perfectly well defined. We can extend this notion to make sense of $V^{\otimes n|m}$ via the parity reversing functor Π. Define

$$V^{n|m} := \underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \times \underbrace{\Pi(V) \times \Pi(V) \times \cdots \times \Pi(V)}_{m \text{ times}},$$

from which the definition of $V^{\otimes n|m}$ follows by the universal property. In other words, we have:

$$V^{\otimes n|m} := V \otimes V \otimes \cdots \otimes V \otimes \Pi(V) \otimes \Pi(V) \otimes \cdots \otimes \Pi(V)$$

where the parity is coming from the tensor product.

In the ordinary setting, an algebra is a vector space A with a multiplication which is bilinear. We may therefore think of it as a vector space A together with a linear map $A \otimes A \to A$. We now define a superalgebra in the same way:

Definition 1.1.7. A *superalgebra* is a super vector space A together with a multiplication morphism $\tau \colon A \otimes A \to A$.

We then say that a superalgebra A is (*super*)*commutative* if

$$\tau \circ c_{A,A} = \tau,$$

that is, if the product of homogeneous elements obeys the rule

$$ab = (-1)^{|a||b|} ba.$$

This is an example of the sign rule mentioned earlier. Note that the signs do not appear in the definition; this is the advantage of the categorical view point which suppresses signs and therefore streamlines the theory.

Similarly we say that A is *associative* if

$$\tau \circ \tau \otimes \mathrm{id} = \tau \circ \mathrm{id} \otimes \tau$$

on $A \otimes A \otimes A$. In other words if $(ab)c = a(bc)$. We also say that A has a *unit* if there is an even element 1 so that

$$\tau(1 \otimes a) = \tau(a \otimes 1) = a$$

for all $a \in A$, that is if $a \cdot 1 = 1 \cdot a = a$.

The tensor product $A \otimes B$ of two superalgebras A and B is again a superalgebra, with multiplication defined as

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd).$$

As an example of associative superalgebra we are going to define the tensor superalgebra.

Definition 1.1.8. Let V be a super vector space. We define *tensor superalgebra* to be the super vector space

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}, \quad T(V)_0 = \bigoplus_{n \text{ even}} V^{\otimes n}, \quad T(V)_1 = \bigoplus_{n \text{ odd}} V^{\otimes n},$$

together with the product defined, as usual, via the ordinary bilinear map $\phi_{r,s} \colon V^{\otimes r} \times V^{\otimes s} \to V^{\otimes (r+s)}$,

$$\phi_{r,s}(v_{i_1} \otimes \cdots \otimes v_{i_r}, w_{j_1} \otimes \cdots \otimes w_{j_s}) = v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes w_{j_1} \cdots \otimes w_{j_s}.$$

One can check that $T(V)$ is a well-defined associative superalgebra with unit, which is noncommutative except when V is even and one-dimensional.

From now on we will assume that all superalgebras are associative and with unit unless specified. Moreover we shall denote the category of commutative superalgebras by (salg).

If we take a super vector space and mod out the odd part, we obtain a classical (that is, purely even) vector space. In a superalgebra the corresponding object is defined by taking the quotient by the ideal generated by the odd elements. This allows one to always refer back to the classical setting.

We denote by J_A the ideal in the commutative superalgebra A generated by the odd elements in A.

Example 1.1.9 (Grassmann coordinates). Let

$$A = k[t_1, \dots, t_p, \theta_1, \dots, \theta_q]$$

where the $t_1, \dots, t_p$ are ordinary indeterminates and the $\theta_1, \dots, \theta_q$ are *odd indeterminates*, i.e., they behave like Grassmann coordinates:

$$\theta_i \theta_j = -\theta_j \theta_i.$$

(This of course implies that $\theta_i^2 = 0$ for all i.) In other words we can view A as the ordinary tensor product $k[t_1, \dots, t_p] \otimes \wedge(\theta_1, \dots, \theta_q)$, where $\wedge(\theta_1, \dots, \theta_q)$ is the exterior algebra generated by $\theta_1, \dots, \theta_q$.

As one can readily check, A is a supercommutative algebra. In fact,

$$A_0 = \left\{ f_0 + \textstyle\sum_{|I| \text{ even}} f_I \theta_I \mid I = \{i_1 < \cdots < i_r\} \right\}$$

where $\theta_I = \theta_{i_1} \theta_{i_2} \dots \theta_{i_r}$, $|I| = r$ and $f_0, f_I \in k[t_1, \dots, t_p]$, and

$$A_1 = \left\{ \textstyle\sum_{|J| \text{ odd}} f_J \theta_J \mid J = \{j_1 < \cdots < j_s\} \right\}.$$

Note that although the $\{\theta_j\} \in A_1$, there are plenty of nilpotents in A_0; take for example $\theta_1 \theta_2 \in A_0$.

This example is important since any finitely generated commutative superalgebra is isomorphic to a quotient of the algebra A by a homogeneous ideal.

As one can readily check, $J_A = (\theta_1, \dots, \theta_q)$ and $A/J_A \cong k[t_1, \dots, t_p]$.

Another important example of commutative superalgebra is the symmetric algebra over a super vector space V.

Consider the natural action of the permutation group S_n on the tensor product $V^{\otimes n}$:

$$s \cdot v_1 \otimes \cdots \otimes v_n = (-1)^{p(s)} v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)}, \quad s \in S_n,$$

where

$$p(s) := |\{(k,l) \mid k > l,\ v_k, v_l \text{ odd},\ s(k) < s(l)\}|.$$

Let $\mathfrak{a}_n$ be the subspace generated by the elements

$$v_1 \otimes \cdots \otimes v_n - s \cdot v_1 \otimes \cdots \otimes v_n \quad \text{for all } s \in S_n \text{ and all } v_i \in V.$$

Observe that $\mathfrak{a} = \bigoplus_{n \geq 0} \mathfrak{a}_n$ is an ideal in $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$.

Definition 1.1.10. Let V be a super vector space. We define the *symmetric n-power* $\mathrm{Sym}^n(V)$ as the super vector space

$$\mathrm{Sym}^n(V) = V^{\otimes n}/\mathfrak{a}_n.$$

We also define the *symmetric superalgebra* $\mathrm{Sym}(V)$ as the superalgebra

$$\mathrm{Sym}(V) = T(V)/\mathfrak{a}.$$

The next proposition is a simple exercise.

Proposition 1.1.11. *Let V be a super vector space.*

$$\mathrm{Hom}_{(\mathrm{salg})}(\mathrm{Sym}(V), A) = \mathrm{Hom}_{(\mathrm{smod})}(V, A)$$

for any commutative superalgebra A. In other words any linear morphism $V \to A$ extends uniquely to a superalgebra morphism $\mathrm{Sym}(V) \to A$.

In a similar way we can also define the exterior algebra.

Definition 1.1.12. Let V be a super vector space. We define the *exterior n-power* $\bigwedge^n(V)$ as the super vector space

$$\bigwedge^n(V) = V^{\otimes n}/\mathfrak{b}_n,$$

where $\mathfrak{b}_n$ is the subspace of $V^{\otimes n}$ generated by the elements

$$v_1 \otimes \cdots \otimes v_n - (-1)^{p(s)} s \cdot v_1 \otimes \cdots \otimes v_n \quad \text{for all } s \in S_n \text{ and all } v_i \in V,$$

where $p(s)$ denotes again the parity of the permutation s. We also define the *exterior superalgebra* $\bigwedge(V)$ as the superalgebra

$$\bigwedge(V) = T(V)/\mathfrak{b},$$

with $\mathfrak{b} = \bigoplus_{n \geq 0} \mathfrak{b}_n$.

Example 1.1.13. If $V = k^{p|q}$,

$$\mathrm{Sym}(V) = k[t_1, \ldots, t_p] \otimes \bigwedge[s_1, \ldots, s_q],$$

$$\bigwedge(V) = k[s_1, \ldots, s_q] \otimes \bigwedge[t_1, \ldots, t_p],$$

where the t_i and s_j are just (even) indeterminates.

Observation 1.1.14. We have the following:

(1) $\mathrm{Sym}(V) \cong \mathrm{Sym}(V_0) \otimes \bigwedge(V_1)$, where $\bigwedge(V_1)$ denotes in this case just the usual exterior algebra of the ordinary vector space V_1.

(2) $\bigwedge(V) \cong \bigwedge(V_0) \otimes \mathrm{Sym}(V_1)$, where $\mathrm{Sym}(V_1)$ denotes in this case just the usual symmetric algebra of the ordinary vector space V_1.

(3) $\mathrm{Sym}(\Pi V) = \bigwedge(V)$, $\bigwedge(\Pi(V)) = \mathrm{Sym}(V)$.

Our next goal is to define derivations in the super context. An *even derivation* of a superalgebra A is a super vector space homomorphism $D \colon A \to A$ such that for $a, b \in A$, $D(ab) = D(a)b + aD(b)$. We may of course extend this definition to include odd linear maps:

Definition 1.1.15. Let $D \in \underline{\mathrm{Hom}}_k(A, A)$ be a k-linear map. Then D is a *derivation* of the superalgebra A if

$$D(ab) = D(a)b + (-1)^{|D||a|} a D(b) \tag{1.1}$$

for $a, b \in A$.

This definition itself is an instance of the sign rule; since a and D are being interchanged, the sign factor appears.

The derivations in $\mathrm{Hom}_{(\mathrm{smod})}(A, A)$ are even (as above) while those contained in $\underline{\mathrm{Hom}}_{(\mathrm{smod})}(A, A)_1$ are odd. The set of all derivations of a superalgebra A, denoted $\mathrm{Der}(A)$, is itself a special type of superalgebra called a *super Lie algebra* which we describe in the following section.

Example 1.1.16. Consider the k-linear operators $\{\partial/\partial t_i\}$ and $\{\partial/\partial\theta_j\}$ of the polynomial superalgebra $A = k[t_1, \ldots, t_p, \theta_1, \ldots, \theta_q]$ to itself defined as

$$\partial/\partial t_i(t_k) = \delta^i_k, \quad \partial/\partial t_i(\theta_l) = 0,$$
$$\partial/\partial\theta_j(t_k) = 0, \quad \partial/\partial\theta_j(\theta_l) = \delta^j_l,$$

and extended to the whole of A according to (1.1). Then $\{\partial/\partial t_i, \partial/\partial\theta_j\} \in \mathrm{Der}(A)$, and we have

$$\mathrm{Der}(A) = \mathrm{span}_A \left\{ \frac{\partial}{\partial t_i}, \frac{\partial}{\partial\theta_j} \right\}.$$

$\partial/\partial\theta_i$ is to be viewed as an odd derivation.

1.2 Super Lie algebras

An important object in supersymmetry is the super Lie algebra.

Definition 1.2.1. A *super Lie algebra* L is an object in the category of super vector spaces together with a morphism $[\ ,\ ]\colon L \otimes L \to L$, often called the *super bracket*, or simply, the *bracket*, which satisfies the following conditions.

(1) Anti-symmetry

$$[\ ,\ ] + [\ ,\ ] \circ c_{L,L} = 0$$

which is the same as $[x, y] + (-1)^{|x||y|}[y, x] = 0$ for $x, y \in L$ homogeneous.

(2) The Jacobi identity

$$[\ ,[\ ,\ ]] + [\ ,[\ ,\ ]] \circ \sigma + [\ ,[\ ,\ ]] \circ \sigma^2 = 0$$

where $\sigma \in S_3$ is a three-cycle, i.e., it takes the first entry of $[,[\ ,\ ]]$ to the second, the second to the third, and the third to the first. So for $x, y, z \in L$ homogeneous, this reads:

$$[x,[y,z]] + (-1)^{|x||y|+|x||z|}[y,[z,x]] + (-1)^{|y||z|+|x||z|}[z,[x,y]] = 0.$$

It is important to note that in the super category, these conditions are modifications of the properties of the bracket in a Lie algebra, designed to accommodate the odd variables.

We shall often use also the term super Lie algebra instead of Lie superalgebra since both are present in the literature.

Remark 1.2.2. We can immediately extend this definition to the case where L is an A-module for A a commutative superalgebra, thus defining a Lie superalgebra in the category of A-modules that we shall discuss in detail in Section 1.3.

Example 1.2.3. Define the associative superalgebra $\mathrm{End}(V)$ as the super vector space $\underline{\mathrm{Hom}}(V,V)$:

$$\mathrm{End}(V) = \underline{\mathrm{Hom}}(V,V)_0 \oplus \underline{\mathrm{Hom}}(V,V)_1$$

with the composition as product. It is a Lie superalgebra with bracket

$$[X,Y] = XY - (-1)^{|X||Y|}YX,$$

where the bracket as usual is defined only on homogeneous elements and then extended by linearity.

Example 1.2.4. In Example 1.1.16 above,

$$\mathrm{Der}(A) = \mathrm{span}_A \left\{ \frac{\partial}{\partial t_i}, \frac{\partial}{\partial \theta_j} \right\}$$

is a super Lie algebra where the bracket is defined, for $D_1, D_2 \in \mathrm{Der}(A)$, to be $[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$.

In fact, we can make any associative superalgebra A into a Lie superalgebra by taking the bracket to be

$$[a, b] = ab - (-1)^{|a||b|} ba,$$

i.e., we take the bracket to be the difference $\tau - \tau \circ c_{A,A}$ where we recall τ is the multiplication morphism on A. We will discuss other examples of super Lie algebras after the following discussion of superalgebra modules. In particular we want to examine the super version of a matrix algebra.

Remark 1.2.5. If the ground field has characteristic 2 or 3 in addition to the antisymmetry and Jacobi conditions, one requires that $[x, x] = 0$ for x even if the characteristic is 2 and $[y, [y, y]] = 0$ for y odd if the characteristic is 3. For more details on superalgebras over fields with positive characteristic see [3].

1.3 Modules for superalgebras

Let A be a superalgebra, associative, but not necessarily commutative, in this section.

Definition 1.3.1. A *left A-module* is a super vector space M with a morphism $A \otimes M \to M$, $a \otimes m \mapsto am$ of super vector spaces obeying the usual identities; that is, for all a, $b \in A$ and x, $y \in M$ we have

(1) $a(x + y) = ax + ay$,

(2) $(a + b)x = ax + bx$,

(3) $(ab)x = a(bx)$,

(4) $1x = x$.

A *right A-module* is defined similarly. Note that if A is commutative, a left A-module is also a right A-module if we define (the sign rule)

$$m \cdot a = (-1)^{|m||a|} a \cdot m$$

for $m \in M$, $a \in A$. Morphisms of A-modules are defined in the obvious manner: they are super vector space morphisms $\phi\colon M \to N$ such that $\phi(am) = a\phi(m)$ for all $a \in A$, $m \in M$. So we have the category of A-modules. For A commutative, the category of A-modules admits tensor products: for M_1, M_2 A-modules, $M_1 \otimes M_2$ is taken as the tensor product of M_1 as a right module with M_2 as a left module.

Let us now turn our attention to *free* A-modules. We already have the notion of the super vector space $k^{p|q}$ over k, and so we define $A^{p|q} := A \otimes k^{p|q}$ where

$$(A^{p|q})_0 = A_0 \otimes (k^{p|q})_0 \oplus A_1 \otimes (k^{p|q})_1,$$
$$(A^{p|q})_1 = A_1 \otimes (k^{p|q})_0 \oplus A_0. \otimes (k^{p|q})_1.$$

Definition 1.3.2. We say that an A-module M is *free* if it is isomorphic (in the category of A-modules) to $A^{p|q}$ for some (p, q).

This definition is equivalent to saying that M contains p *even* elements $\{e_1, \ldots, e_p\}$ and q *odd* elements $\{\epsilon_1, \ldots, \epsilon_q\}$ such that

$$M_0 = \operatorname{span}_{A_0}\{e_1, \ldots, e_p\} \oplus \operatorname{span}_{A_1}\{\epsilon_1, \ldots, \epsilon_q\},$$
$$M_1 = \operatorname{span}_{A_1}\{e_1, \ldots, e_p\} \oplus \operatorname{span}_{A_0}\{\epsilon_1, \ldots, \epsilon_q\}.$$

We shall also say that M is the *free module* generated over A by the even elements $e_1, \ldots, e_p$ and the odd elements $\epsilon_1, \ldots, \epsilon_q$.

Let $T: A^{p|q} \to A^{r|s}$ be a morphism of free A-modules and write $e_{p+1}, \ldots, e_{p+q}$ for the odd basis elements $\epsilon_1, \ldots, \epsilon_q$. Then T is defined on the basis elements $\{e_1, \ldots, e_{p+q}\}$ by

$$T(e_j) = \sum_{i=1}^{p+q} e_i t^i_j. \tag{1.2}$$

Hence T can be represented as a matrix of size $(r + s) \times (p + q)$:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{1.3}$$

where T_1 is an $r \times p$ matrix consisting of even elements of A, T_2 is an $r \times q$ matrix of odd elements, T_3 is an $s \times p$ matrix of odd elements, and T_4 is an $s \times q$ matrix of even elements. We say that T_1 and T_4 are *even blocks* and that T_2 and T_3 are *odd blocks*. Note that the fact that T is a morphism of super A-modules means that it must preserve parity, and therefore the parity of the blocks is determined. Note also that when we define T on the basis elements, in the expression (1.2) the basis element *precedes* the coordinates t^i_j. This is important to keep the signs in order and comes naturally from composing morphisms. In other words if the module is written as a right module with T acting from the left, composition becomes matrix product in the usual manner:

$$(S \cdot T)(e_j) = S\Big(\sum_i e_i t^i_j\Big) = \sum_{i,k} e_k s^k_i t^i_j.$$

Hence for any $x \in A^{p|q}$, we can express x as the column vector $x = \sum e_i x^i$ and so $T(x)$ is given by the matrix product Tx.

1.4 The language of matrices

Let A be a commutative superalgebra.

Let us now consider all endomorphisms of $M = A^{p|q}$, i.e., $\operatorname{Hom}(M, M)$. This is an ordinary algebra (i.e., *not* super) of matrices of the same type as T above. Even though in matrix form each morphism contains blocks of odd elements of A, each morphism is an even linear map from M to itself since a morphism in the super category must preserve parity. In order to get a truly supergeometric version of the ordinary matrix algebra, we must consider *all* linear maps M to M, i.e., we are interested in $\underline{\mathrm{Hom}}(M, M)$. Now we can talk about even and odd matrices. An even matrix T takes on the block form from above. But the parity of the blocks is reversed for an odd matrix S; we get

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

where S_1 is a $p \times p$ odd block, S_4 is a $q \times q$ odd block, S_2 is a $p \times q$ even block, and S_3 is a $q \times p$ even block. Note that in the case where $M = k^{p|q}$, the odd blocks are just zero blocks. We will denote this superalgebra of even and odd $(p+q) \times (p+q) = p|q \times p|q$ matrices by $\mathrm{Mat}(A^{p|q})$. This superalgebra is in fact a super Lie algebra where we define the bracket [,] as in Example 1.2.4:

$$[T, S] = TS - (-1)^{|T||S|} ST$$

for $S, T \in \mathrm{Mat}(A^{p|q})$.

Remark 1.4.1. Note that $\mathrm{Mat}(A^{p|q}) = \underline{\mathrm{Hom}}(A^{p|q}, A^{p|q})$. We do not want to confuse this with what we will later denote as $\mathrm{M}_{p|q}(A)$, which will functorially only include the *even part* of $\mathrm{Mat}(A^{p|q})$, i.e.,

$$\mathrm{Mat}(A^{p|q})_0 = \mathrm{M}_{p|q}(A) = \mathrm{Hom}(A^{p|q}, A^{p|q}).$$

We shall come back to this key model in Example 3.1.5.

We now turn to the supergeometric extensions of the trace and determinant. Let $T : A^{p|q} \to A^{p|q}$ be a morphism (i.e., $T \in (\mathrm{Mat}(A^{p|q}))_0$) with block form (1.3).

Definition 1.4.2. We define the *super trace* of T to be

$$\mathrm{str}(T) := \mathrm{tr}(T_1) - \mathrm{tr}(T_4),$$

where "tr" denotes the ordinary trace.

This negative sign is actually forced upon us when we take a categorical view of the trace. We will not discuss this here, but we later motivate this definition when we explore the supergeometric-extension of the determinant.

Remark 1.4.3. The super trace is actually defined for *all* linear maps. For $S \in \mathrm{Mat}(A^{p|q})_1$ an odd matrix,

$$\mathrm{str}(S) = \mathrm{tr}(S_1) + \mathrm{tr}(S_4).$$

Note the sign change. Note also that the trace is commutative, meaning that for even matrices $A, B \in \mathrm{Mat}(A^{p|q})_0$, we have the familiar formula

$$\mathrm{str}(AB) = \mathrm{str}(BA)$$

that we shall prove in Observation 1.5.8 after we have introduced the notion of Berezinian.

Definition 1.4.4. If M is an A-module, then $\mathrm{GL}(M)$ is defined as the group of automorphisms of M and we call it the *super general linear group of automorphisms of* M. If $M = A^{p|q}$ the free A-module generated by p even and q odd variables, then we write $\mathrm{GL}(M) = \mathrm{GL}_{p|q}(A)$. We may also use the notation $\mathrm{GL}_{p|q}(A) = \mathrm{GL}(A^{p|q})$.

1.5 The Berezinian

We want to define the generalization of the determinant, called the *Berezinian*, on elements of $\mathrm{GL}(A^{p|q})$. We may say that this is the point where linear supergeometry differs most dramatically from the ordinary theory.

Proposition 1.5.1. *Let $T: A^{p|q} \to A^{p|q}$ be a morphism with the usual block form* (1.3)*. Then T is invertible if and only if T_1 and T_4 are invertible.*

Proof. Let $J_A \subset A$ be the ideal generated by the odd elements and let $\bar{A} = A/J_A$. There is a natural map $\mathrm{M}_{p|q}(A) \to \mathrm{M}_{p|q}(\bar{A})$, $T \mapsto \bar{T}$, where $\bar{T}$ is obtained from the matrix T by applying to its entries the map $A \to \bar{A}$. We claim that T is invertible if and only if $\bar{T}$ is invertible. One direction is obvious, namely the case in which T is invertible. Now assume that $\bar{T}$ is invertible. This implies that there exists $\bar{S} \in \mathrm{M}_{p|q}(\bar{A})$ such that $\bar{T}\bar{S} = \bar{S}\bar{T} = I$, where I denotes the identity (both in $\mathrm{M}_{p|q}(\bar{A})$ and in $\mathrm{M}_{p|q}(A)$). Hence there exists $S \in \mathrm{M}_{p|q}(A)$ such that $TS = I + N$, with $N \in \mathrm{M}_{p|q}(A)$ (we consider only the case of a right inverse since the left inverse is the same). To prove T is invertible it is enough to show that N is nilpotent, i.e., $N^r = 0$ for some r. Since the entries of N^m are in A_1^m for m sufficiently large they are all zero. □

Definition 1.5.2. Let T be an invertible element in $\mathrm{M}_{p|q}(A)$, i.e., $T \in \mathrm{GL}(A^{p|q})$ with the standard block form (1.3) from above. Then we formulate the Berezinian Ber:

$$\mathrm{Ber}(T) = \det(T_1 - T_2 T_4^{-1} T_3) \det(T_4)^{-1} \tag{1.4}$$

where "det" is the usual determinant.

The Berezinian is named after Berezin, who was one of the pioneers of superalgebra and superanalysis.

Remark 1.5.3. The first thing we notice is that in the super category, we only define the Berezinian for *invertible* transformations. This marks an important difference with the determinant, which is defined in ordinary linear algebra for all endomorphisms of a vector space. We immediately see that it is necessary that the block T_4 be invertible for the formula (1.4) to make sense, however one can actually define the Berezinian on all matrices with *only* the T_4 block invertible (i.e., the matrix itself may not be invertible, but the T_4 block is). There is a similar formulation of the Berezinian which requires that only the T_1 block be invertible:

$$\mathrm{Ber}(T) = \det(T_4 - T_3 T_1^{-1} T_2)^{-1} \det(T_1).$$

So we can actually define the Berezinian on all matrices with *either* the T_1 *or* the T_4 block invertible. Note that in the case where both blocks are invertible (i.e., when the matrix T is invertible), both formulae of the Berezinian give the same answer as we shall see after the next proposition.

Proposition 1.5.4. *The Berezinian is multiplicative: For* $S, T \in \mathrm{GL}(A^{p|q})$*,*

$$\mathrm{Ber}(ST) = \mathrm{Ber}(S)\,\mathrm{Ber}(T).$$

Proof. We will only briefly sketch the proof here and leave the details to the reader. First note that any $T \in \mathrm{GL}(A^{p|q})$ with block form (1.3) may be written as the product of the following "elementary matrices":

$$T_+ = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}, \quad T_0 = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix}.$$

If we equate $T = T_+ T_0 T_-$, we get a system of equations which lead to the solution

$$\begin{aligned} X &= T_2 T_4^{-1}, \\ Y_1 &= T_1 - T_2 T_4^{-1} T_3, \\ Y_2 &= T_4, \\ Z &= T_4^{-1} T_3. \end{aligned}$$

It is also easy to verify that $\mathrm{Ber}(ST) = \mathrm{Ber}(S)\,\mathrm{Ber}(T)$ for S of type $\{T_+, T_0\}$ for all T *or* T of type $\{T_-, T_0\}$ for all S. Let $G \subset \mathrm{GL}_{p|q}(A)$ be the set of elements S such that $\mathrm{Ber}(ST) = \mathrm{Ber}(S)\,\mathrm{Ber}(T)$ for all T. One can check right away that G is a subgroup of $\mathrm{GL}_{p|q}(A)$. To prove our result is it enough to show that matrices of type T_+, T_-, $T_0 \in G$ since they generate $\mathrm{GL}_{p|q}(A)$. By our previous discussion T_+, $T_0 \in G$, hence we only need to show that $\mathrm{Ber}(ST) = \mathrm{Ber}(S)\,\mathrm{Ber}(T)$ for S of type T_- for all T. Notice that

$$\mathrm{Ber}(ST_+T_0T_-) = \mathrm{Ber}(ST_+)\,\mathrm{Ber}(T_0)\,\mathrm{Ber}(T_-)$$

as we have already seen. Hence, the last case to verify is for

$$S = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}.$$

We may assume that both X and Z each have only one non-zero entry since the product of two matrices of type T_+ results in the sum of the upper right blocks, and likewise with the product of two type T_- matrices. Let $x_{ij}, z_{kl} \neq 0$. Then

$$ST = \begin{pmatrix} 1 & X \\ Z & 1 + ZX \end{pmatrix}$$

and $\mathrm{Ber}(ST) = \det(1 - X(1 + ZX)^{-1}Z)\det(1 + ZX)^{-1}$. Since both X and Z have only one non-zero entry, $(1 + ZX)^{-1} = (1 - ZX)$, hence

$$\det(1 - X(1 + ZX)^{-1}Z) = \det(1 - X(1 - ZX)Z) = \det(1 - XZ).$$

This is because all the values within the determinants are either upper triangular or contain an entire column of zeros (X, Z have at most one non-zero entry), the values x_{ij}, z_{kl} contribute to the determinant only when the product ZX has its non-zero term on the diagonal, i.e., only when $i = j = k = l$. $\mathrm{Ber}(ST) = \det(1 - XZ)\det(1 - ZX) = (1 - x_{ii}z_{ii})(1 + x_{ii}z_{ii}) = 1$. A direct calculation shows that $\mathrm{Ber}(S) = \mathrm{Ber}(T) = 1$. □

Corollary 1.5.5. *Let $T \in \mathrm{GL}_{p|q}(A)$. Then*

$$\mathrm{Ber}(T) = \det(T_4 - T_3T_1^{-1}T_2)^{-1}\det(T_1).$$

Proof. Consider the decomposition

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T_3T_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_4 - T_3T_1^{-1}T_2 \end{pmatrix} \begin{pmatrix} 1 & T_1^{-1}T_2 \\ 0 & 1 \end{pmatrix}.$$

By multiplicativity of the Berezinian, we obtain the result. □

Corollary 1.5.6. *The Berezinian is a homomorphism*

$$\mathrm{Ber}\colon \mathrm{GL}(A^{p|q}) \to \mathrm{GL}_{1|0}(A) = A_0^{\times}$$

into the invertible elements of A.

Proof. This follows immediately from the multiplicativity property. □

Remark 1.5.7. In the course of the proof of Corollary 1.5.5 we have seen the decomposition

$$\mathrm{GL}_{p|q}(A) = UHV,$$

where

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \right\}, \quad V = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \right\}.$$

This very much resembles the big cell decomposition in the theory of ordinary algebraic groups, however here the decomposition holds *globally*.

Observation 1.5.8. The usual determinant on the general linear group GL_n induces the trace on its Lie algebra, namely the matrices M_n (see Remark 1.4.1). Let ϵ be an even indeterminate, $\epsilon^2 = 0$ and let us compute the value $\mathrm{Ber}(1 + \epsilon T) \in A[\epsilon]$:

$$\mathrm{Ber}(1 + \epsilon T) = \det(1 + \epsilon T_1)\det(1 - \epsilon T_4) = (1 + \epsilon\,\mathrm{tr}\,T_1)(1 - \epsilon\,\mathrm{tr}\,T_4) = 1 + \epsilon\,\mathrm{str}(T).$$

We are now going to see that $\mathrm{str}(ST) = \mathrm{str}(TS)$.

$$\begin{aligned}\mathrm{Ber}(1+\epsilon ST) &= (1+\epsilon\,\mathrm{tr}(ST)_1)(1-\epsilon\,\mathrm{tr}(ST)_4)\\ &= (1+\epsilon\,\mathrm{tr}(S_1T_1+S_2T_3))(1-\epsilon\,\mathrm{tr}(S_3T_2+S_4T_4))\\ &= (1+\epsilon\,\mathrm{tr}(T_1S_1)-\epsilon\,\mathrm{tr}(T_3S_2))(1+\epsilon\,\mathrm{tr}(T_2S_3)-\epsilon\,\mathrm{tr}(T_4S_4))\\ &= 1+\epsilon\,\mathrm{tr}(T_1S_1+T_2S_3-T_3S_2-T_4S_4)\\ &= 1+\epsilon\,\mathrm{str}(TS),\end{aligned}$$

Hence

$$\mathrm{Ber}(1+\epsilon ST) = 1+\epsilon\,\mathrm{str}(ST) = 1+\epsilon\,\mathrm{str}(TS),$$

thus proving our claim.

This of course leads to the question of how the formula for the Berezinian arises. The answer lies in the supergeometric version of integral forms on supermanifolds called *densities*. In his pioneering work in superanalysis, F. A. Berezin calculated the change of variables formula for densities on isomorphic open submanifolds of $\mathbb{R}^{p|q}$ ([10]). This led to an extension of the Jacobian in ordinary differential geometry; the Berezinian is so-named after him.

We are ready for the formula for the inverse of a supermatrix.

Proposition 1.5.9. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathrm{GL}_{p|q}(A)$$

(*hence T_1 and T_4 are invertible ordinary matrices*). *Then*

$$T^{-1} = \begin{pmatrix} (T_1 - T_2T_4^{-1}T_3)^{-1} & -T_1^{-1}T_2(T_4 - T_3T_1^{-1}T_2)^{-1} \\ -T_4^{-1}T_3(T_1 - T_2T_4^{-1}T_3)^{-1} & (T_4 - T_3T_1^{-1}T_2)^{-1} \end{pmatrix}.$$

Proof. Direct check. □

We finish our summary of superlinear algebra by giving meaning to the *rank* of an endomorphism of $A^{p|q}$. We shall return to this concept and an important variation of it in Chapter 5, Section 5.2.

Definition 1.5.10. Let $T \in \mathrm{End}(A^{p|q})$. Then the *rank* of T, $\mathrm{rank}(T)$, is the superdimension of the largest invertible submatrix of T (obtained by removing columns and rows).

Proposition 1.5.11. *Again, let $T \in \mathrm{End}(A^{p|q})$ with block form* (1.3). *Then* $\mathrm{rank}(T) = \mathrm{rank}(T_1)|\,\mathrm{rank}(T_4)$.

Proof. Assume that $\mathrm{rank}(T) = r|s$. Then there is an invertible $r|s \times r|s$ submatrix of T and it is clear that $r \leq \mathrm{rank}(T_1)$, $s \leq \mathrm{rank}(T_4)$. Conversely, if $\mathrm{rank}(T_1) = r'$, $\mathrm{rank}(T_4) = s'$ it is also clear that there exists an invertible $r'|s' \times r'|s'$ submatrix of T. Therefore we must have $r = r'$, $s = s'$. □

1.6 The universal enveloping superalgebra

Given a Lie superalgebra $\mathfrak{g}$, possibly infinite-dimensional, we would like to introduce an associative superalgebra $\mathfrak{U}(\mathfrak{g})$, called the *universal enveloping superalgebra* of $\mathfrak{g}$ (UESA for short) with a natural universal property with respect to the Lie superalgebra structure of $\mathfrak{g}$. The universal enveloping superalgebra is the main tool to convert Lie superalgebra problems into associative superalgebra ones. For example we will see that a representation of $\mathfrak{g}$ can be uniquely extended to a representation of its UESA $\mathfrak{U}(\mathfrak{g})$. The main results regarding the UESA are the existence and uniqueness of the UESA and the Poincaré–Birkhoff–Witt Theorem and we will discuss them in full detail. In order to keep the exposition simple, we shall treat the case when $\mathfrak{g}$ has a linearly ordered basis (e.g., when it is countable dimensional) as most applications fall in this framework. By the well-ordering principle this hypothesis is unnecessary, but we shall not press the point.

Definition 1.6.1. Let $\mathfrak{g}$ be a Lie superalgebra, $T(\mathfrak{g})$ the tensor superalgebra over the underlying super vector space of $\mathfrak{g}$, $i : \mathfrak{g} \to T(\mathfrak{g})$ the natural immersion. Let $I \subset T(\mathfrak{g})$ be the two-sided ideal in $T(\mathfrak{g})$ generated by

$$i(X) \otimes i(Y) - (-1)^{|X||Y|} i(Y) \otimes i(X) - i([X,Y]) \in T(\mathfrak{g}) \quad \text{for all } X, Y \in \mathfrak{g}.$$

(As usual in the super setting we give relations only for homogeneous elements.)

We define $\mathfrak{U}(\mathfrak{g})$ the *universal enveloping superalgebra* (UESA) of $\mathfrak{g}$ as

$$\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g})/I.$$

We denote the product of two elements $X, Y \in \mathfrak{U}(\mathfrak{g})$ by XY.

As any associative superalgebra, $\mathfrak{U}(\mathfrak{g})$ has a natural Lie superalgebra structure: $[X,Y] = XY - (-1)^{|X||Y|} YX$.

Let $j : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ be the composition of the injective linear map $i : \mathfrak{g} \to T(\mathfrak{g})$ and the surjective superalgebra morphism $\pi : T(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I$. We shall see later that this j is injective.

We are now ready to state the universal property of $\mathfrak{U}(\mathfrak{g})$.

Proposition 1.6.2. *The UESA* $\mathfrak{U}(\mathfrak{g})$ *has the following properties:*

(1) $j(\mathfrak{g})$ *generates* $\mathfrak{U}(\mathfrak{g})$ (*as an associative superalgebra*).

(2) $j([X,Y]) = j(X)j(Y) - (-1)^{|X||Y|} j(Y)j(X)$; *in other words,* j *is a Lie superalgebra morphism.*

(3) *If* A *is any associative superalgebra and* $\xi : \mathfrak{g} \to A$ *is a Lie superalgebra morphism, i.e.,* $\xi([X,Y]) = \xi(X)\xi(Y) - (-1)^{|X||Y|}\xi(Y)\xi(X)$*, then there exists a unique morphism* σ *of associative superalgebras such that the following diagram*

is commutative:

$$\begin{array}{ccc} & & \mathfrak{U}(\mathfrak{g}) \\ & \overset{j}{\nearrow} & \downarrow \sigma \\ \mathfrak{g} & \xrightarrow{\xi} & A. \end{array}$$

Proof. (1) $j(\mathfrak{g})$ generates $\mathfrak{U}(\mathfrak{g})$ since $j = \pi \circ i$ and $i(\mathfrak{g})$ generates $T(\mathfrak{g})$.

(2) Since $\pi(I) = 0$,

$$\begin{aligned} & j([X,Y]) - j(X)j(Y) + (-1)^{|X||Y|} j(Y)j(X) \\ & \qquad = \pi(i([X,Y]) - i(X)i(Y) + (-1)^{|X||Y|} i(Y)i(X)) = 0. \end{aligned}$$

(3) Define the superalgebra morphism $\xi' \colon T(\mathfrak{g}) \to A$, $\xi'(i(X)) = \xi(X)$ for all $X \in \mathfrak{g}$. Since by definition $\xi'(I) = 0$ we have that ξ' factors as $\sigma \circ \pi$ for some $\sigma \colon \mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I \to A$. The uniqueness of σ comes from (1). □

We give now an important application of this result: we prove that a representation of a Lie superalgebra extends uniquely to a representation of its UESA.

Definition 1.6.3. We define a *representation* of a Lie superalgebra $\mathfrak{g}$ in a super vector space V, as a linear morphism $\rho \colon \mathfrak{g} \to \mathrm{End}(V)$ preserving parity and the bracket. We say also that V is a $\mathfrak{g}$*-module*. A $\mathfrak{g}$-module V is *irreducible* if it does not have non-trivial submodules. We also define a *representation* of an associative superalgebra A in V as a superalgebra morphism $\rho' \colon A \to \mathrm{End}(V)$.

Theorem 1.6.4. *Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{U}(\mathfrak{g})$ its UESA. Let ρ be a representation of $\mathfrak{g}$ in a super vector space V. Then ρ extends uniquely to a representation ρ' of $\mathfrak{U}(\mathfrak{g})$ in V.*

Proof. Take $A = \mathrm{End}(V)$ in Proposition 1.6.2. In fact a representation of $\mathfrak{g}$ in V is a morphism of Lie algebras $\rho \colon \mathfrak{g} \to \mathrm{End}(V)$. By the universality, ρ extends uniquely to a representation $\rho' \colon \mathfrak{U}(\mathfrak{g}) \to \mathrm{End}(V)$. □

We turn now to the Poincaré–Birkhoff–Witt Theorem, which allows us to construct explicitly a basis for the UESA $\mathfrak{U}(\mathfrak{g})$ starting from a basis of $\mathfrak{g}$.

Let $\{X_k, k \in K\}$ be a homogeneous basis of $\mathfrak{g}$, K a linearly ordered set. To simplify the proof and the notation we assume without loss of generality that the indices corresponding to even elements are smaller, in the given order, than the indices corresponding to the odd elements. Moreover we will use the same symbol to denote an element in $\mathfrak{g}$ and its image via i in $T(\mathfrak{g})$.

$\mathfrak{U}(\mathfrak{g})$ is spanned by 1 and $j(X_{k_1}), \dots, j(X_{k_t})$, for all $k_1, \dots, k_t \in K$. Since

$$j(X_k)j(X_l) = (-1)^{p(X_k)p(X_l)} j(X_l)j(X_k) + [j(X_k), j(X_l)]$$

for X_k, $X_l \in \mathfrak{g}$ we expect $\mathfrak{U}(\mathfrak{g})$ to be spanned by

$$1, \quad j(X_{k_1}) \dots j(X_{k_r}) j(X_{k_{r+1}}) \dots j(X_{k_{r+s}}),$$
$$k_1 \le \dots \le k_r < k_{r+1} < \dots < k_{r+s},$$
$$X_{k_i} \text{ even}, \quad 1 \le i \le r, \quad X_{k_j} \text{ odd}, \quad r+1 \le j \le r+s.$$

The fact that an odd element X appears only with exponent 1 is due to the relation $j([X, X]) = 2j(X)^2$. The Poincaré–Birkhoff–Witt Theorem says that these monomials form a basis for $\mathfrak{U}(\mathfrak{g})$.

Let $T^0 \subset T(\mathfrak{g})$ denote the linear super space of the *standard monomials*, that is, the monomials of the form

$$X_{k_1} \otimes \dots \otimes X_{k_r} \otimes X_{k_{r+1}} \otimes \dots \otimes X_{k_{r+s}},$$
$$k_1 \le \dots \le k_r < k_{r+1} < \dots < k_{r+s},$$
$$X_{k_i} \text{ even}, \quad 1 \le i \le r, \quad X_{k_j} \text{ odd}, \quad r+1 \le j \le r+s.$$

Theorem 1.6.5 (Poincaré–Birkhoff–Witt). *Let* $\mathfrak{g}$ *be a Lie superalgebra,* $\mathfrak{U}(\mathfrak{g})$ *its UESA,* $\{X_k, k \in K\}$ *a homogeneous basis of* $\mathfrak{g}$*,* K *a linearly ordered set as above. Then*

$$1, \quad j(X_{k_1}) \dots j(X_{k_r}) j(X_{k_{r+1}}) \dots j(X_{k_{r+s}}),$$
$$k_1 \le \dots \le k_r < k_{r+1} < \dots < k_{r+s},$$
$$X_{k_i} \textit{ even}, \quad 1 \le i \le r, \quad X_{k_j} \textit{ odd}, \quad r+1 \le j \le r+s$$

is a basis for $\mathfrak{U}(\mathfrak{g})$.

Proof. The proof is very similar to the one in the ordinary setting. The only important difference comes from the fact that in a Lie superalgebra for X odd $[X, X]$ may not be zero, hence the relation holding in the UESA $2X^2 - [X, X]$ will have to be examined separately.

The statement of the theorem is equivalent to the two statements:

(1) $I + T^0 = T(\mathfrak{g})$.

(2) $I \cap T^0 = (0)$.

(1) Let T_m denote the span of the monomial tensors of degree m and $T_m^0 = T^0 \cap T_m$. It is enough to prove that

$$T_m \subset I + \sum_{0 \le q \le m} T_q^0.$$

We do this by induction on m. The cases $m = 0, 1$ are clear. If $t = X_{k_1} \otimes \dots \otimes X_{k_u}$, we denote by $\mathrm{ind}(t)$ the number of pairs (p, q) with $1 \le p, q \le u$ such that $p < q$, but $k_p \ge k_q$ if X_{k_p} and X_{k_q} are both odd, $p < q$ but $k_p > k_q$ otherwise. Clearly the elements $t \in T^0$ have $\mathrm{ind}(t) = 0$. Let $T_m^d = \{t \in T_m \mid \mathrm{ind}(t) = d\}$. It is enough to show that $T_m^d \subset I + \sum_{0 \le n \le m} T_n^0$. We use induction on d. The case $d = 0$ is clear. Fix $t = X_{k_1} \otimes \cdots \otimes X_{k_m} \in T_m^d$. If $d > 0$, there exists v, $1 \leq v \leq m-1$, such that $k_v \geq k_{v+1}$, if X_{k_v} and $X_{k_{v+1}}$ are odd, $k_v > k_{v+1}$ otherwise. Assume first that $X_{k_v} \neq X_{k_{v+1}}$. Let $t' = X_{l_1} \otimes \cdots \otimes X_{l_m}$ be defined as follows: $l_i = k_i$ for all $i \neq v, v+1$ and $l_v = k_{v+1}$, $l_{v+1} = k_v$. Then $t' \in T_m^{d-1} \subset I + T^0$, by induction hypothesis. Since

$$X_{k_v} \otimes X_{k_{v+1}} - (-1)^{p(X_{k_v})p(X_{k_{v+1}})} X_{k_{v+1}} \otimes X_{k_v} = [X_{k_v}, X_{k_{v+1}}] \mod I$$

we have $t - t' \in I + T_{m-1} \subset I + \sum_{1 \leq n \leq m-1} T_n^0$, which by induction concludes our argument in case $X_{k_v} \neq X_{k_{v+1}}$. Assume now that $X_{k_v} = X_{k_{v+1}}$ odd. Since $X_{k_v} \otimes X_{k_v} = (1/2)[X_{k_v}, X_{k_v}] \mod I$, we have that $t \in I + T_{m-1} \subset I + \sum_{1 \leq n \leq m-1} T_n^0$, which again by induction concludes our argument.

(2) Our strategy is the following. We shall construct an endomorphism $L \colon T(\mathfrak{g}) \to T(\mathfrak{g})$ such that:

(i) $L(t) = t$ for all standard monomials.

(ii) If $1 \leq v \leq p-1$, $p \geq 2$, $k_v > k_{v+1}$, then

$$\begin{aligned} &L(X_{k_1} \otimes \cdots \otimes X_{k_v} \otimes X_{k_{v+1}} \otimes \cdots \otimes X_{k_p}) \\ &\quad = (-1)^{|X_{k_v}||X_{k_{v+1}}|} L(X_{k_1} \otimes \cdots \otimes X_{k_{v+1}} \otimes X_{k_v} \otimes \cdots \otimes X_{k_p}) \\ &\qquad + L(X_{k_1} \otimes \cdots \otimes [X_{k_v}, X_{k_{v+1}}] \otimes \cdots \otimes X_{k_p}). \end{aligned}$$

If $1 \leq v \leq p-1$, $p \geq 2$, $k_v = k_{v+1}$ and X_{k_v} odd, then

$$\begin{aligned} &L(X_{k_1} \otimes \cdots \otimes X_{k_v} \otimes X_{k_{v+1}} \otimes \cdots \otimes X_{k_p}) \\ &\quad = \tfrac{1}{2} L(X_{k_1} \otimes \cdots \otimes [X_{k_v}, X_{k_{v+1}}] \otimes \cdots \otimes X_{k_p}). \end{aligned}$$

If we succeed in building such an endomorphism we are done since L is the identity on T^0 and it is zero on I. We will define L by induction on p, the degree of the monomials in $T(\mathfrak{g})$. Define L to be the identity on T_0 and T_1. Now assume that $p > 1$. We want to define L on $t = X_{k_1} \otimes \cdots \otimes X_{k_p}$ in such a way that (i) and (ii) are satisfied. We proceed by induction on $d = \operatorname{ind}(t)$. If $d = 0$ we define $L(t) = t$. Assume that $d > 0$ and that we have defined L on every monomial s of degree p and $\operatorname{ind}(s) < d$. Since $d > 0$ there exists an integer r, $1 \leq r \leq p-1$ (not unique) such that $k_r > k_{r+1}$ or $k_r = k_{r+1}$ with X_{k_r} odd. We can then use the right-hand side of (ii) (after choosing the appropriate case) to define $L(t)$ and clearly such L is what we want and is defined by induction on d and on p. We have now to show that such an L is well defined, i.e., the definition is independent from the choice of the integer r. Assume that we first choose the integer r and then another integer l and obtain, following our recipe, a certain expression a for $L(t)$. Then we proceed by first choosing the integer l and then the integer r and obtain another expression b for $L(t)$. We want to show that $a = b$. Whenever $|r - l| \geq 2$ it is easy to convince ourselves that $a = b$, since there will be no "interference" between the two definitions of L. The problems appear when $|r - l| = 1$.

Assume without loss of generality that $r = l - 1$. To ease the notation let $X_{i_r} = X$, $X_{i_{r+1}} = Y$, $X_{i_{r+2}} = Z$, so that we have $t = X_{k_1} \otimes \cdots \otimes X \otimes Y \otimes Z \otimes \cdots \otimes X_{k_p}$. Let us first assume that no two of X, Y, Z are odd and equal. After some calculations one finds that the two possible ways of defining $L(t)$ give the following two results:

$$\begin{aligned} a = &(-1)^{|X||Y|+|Z||X|+|Y||Z|}L(\dots Z \otimes Y \otimes X \dots) \\ &+ (-1)^{|X||Y|+|Z||X|}L(\dots [Y,Z] \otimes X \dots) \\ &+ (-1)^{|X||Y|}L(\dots Y \otimes [X,Z] \dots) + L(\dots [X,Y] \otimes Z \dots), \end{aligned}$$

$$\begin{aligned} b = &(-1)^{|X||Y|+|Z||X|+|Y||Z|}L(\dots Z \otimes Y \otimes X \dots) \\ &+ (-1)^{|Y||Z|+|Z||X|}L(\dots Z \otimes [X,Y] \dots) \\ &+ (-1)^{|Z||Y|}L(\dots [X,Z] \otimes Y \dots) + L(\dots X \otimes [Y,Z] \dots). \end{aligned}$$

Hence we have

$$a - b = L(\cdots - [X,[Y,Z]] + [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]] \dots) = 0$$

because of the Jacobi identity. Hence there is no ambiguity in our definition of L.

Let us now assume that two of X, Y, Z are equal and odd. We can assume without loss of generality to have $t = X_{k_1} \otimes \cdots \otimes X \otimes X \otimes Y \otimes \cdots \otimes X_{k_p}$, the other cases being the same. After some calculations we have

$$\begin{aligned} a = &\tfrac{1}{2}L(\dots Y \otimes [X,X] \dots) + (-1)^{|X||Y|}L(\dots [X,Y] \otimes X \dots) \\ &+ L(\dots X \otimes [X,Y] \dots), \end{aligned}$$

$$b = \tfrac{1}{2}L(\dots Y \otimes [X,X] \dots) + \tfrac{1}{2}L(\dots [[X,X],Y] \dots).$$

Hence

$$a - b = L(\dots [X,[X,Y]] - \tfrac{1}{2}[[X,X],Y] \dots) = 0.$$

Again by the Jacobi identity it follows that $[X,[X,Y]] + (-1)^{|X||Y|}[X,[Y,X]] + [Y,[X,X]] = 2[X,[X,Y]] - [[X,X],Y] = 0$. □

Corollary 1.6.6. *The Lie superalgebra morphism $j : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ described above is an injection.*

To ease the notation, from now on we identify $\mathfrak{g}$ with its image $j(\mathfrak{g})$ in $\mathfrak{U}(\mathfrak{g})$.

Corollary 1.6.7. *We have the linear isomorphism*

$$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1),$$

where $\bigwedge(\mathfrak{g})$ is the exterior algebra over the (ordinary) vector space $\mathfrak{g}_1$.

Next we show that $\mathfrak{U}(\mathfrak{g})$ has a filtered algebra structure and we relate it to $\mathrm{Sym}(\mathfrak{g})$ the symmetric algebra over the super vector space $\mathfrak{g}$.

From now on assume that $\mathfrak{g}$ is finite-dimensional. Let us first recall some basic definitions.

Definition 1.6.8. An associative superalgebra A is *graded* if for all integers $n \geq 0$, we have a subspace $A_{(n)} \subset A$ such that

(1) $1 \in A_{(0)}$,

(2) $A_{(m)} A_{(n)} \subset A_{(m+n)}$,

(3) $A = \bigoplus_{n \geq 0} A_{(n)}$.

The elements in $A_{(n)}$ are said to be $\mathbb{Z}$-*homogeneous of degree* n or just *homogeneous* when there is no ambiguity. We also say that the grading is compatible with the superalgebra structure if $A_0 = \bigoplus_{n \text{ even}} A_{(n)}$, $A_1 = \bigoplus_{n \text{ odd}} A_{(n)}$. All gradings we consider have this property.

We say that an associative superalgebra B is *filtered* if for all integers $n \geq 0$, we have a subspace $B^n \subset A$ such that

(i) $1 \in B^0$,

(ii) $B^n B^m \subset B^{m+n}$,

(iii) $B^0 \subset B^1 \subset \cdots \subset \bigcup_{n \geq 0} B^n = B$.

Any graded superalgebra A can be viewed as a filtered superalgebra defining $A^n = \bigoplus_{0 \leq i \leq n} A_{(i)}$, though not all filtered algebras arise in this way. Vice versa to any filtered superalgebra B, we can associate a graded superalgebra $\mathrm{Gr}(B)$ in the following way. Define $\mathrm{Gr}(B)_{(n)} := B^n / B^{n-1}$ and $\mathrm{Gr}(B) := \bigoplus_{n \geq 0} \mathrm{Gr}(B)_{(n)}$ as vector super spaces. Let $\pi_n \colon B^n \to \mathrm{Gr}(B)_{(n)}$ be the natural projection. Given $a \in \mathrm{Gr}(B)_{(m)}$ and $b \in \mathrm{Gr}(B)_{(n)}$ choose $a' \in B^m$ and $b' \in B^n$ such that $\pi_m(a') = a$ and $\pi_n(b') = b$. Then define the product $ab := \pi_{m+n}(a'b') \in \mathrm{Gr}(B)_{(m+n)}$. One can check this is well defined and that $\mathrm{Gr}(B)$ is a graded superalgebra.

There are two examples of graded and filtered algebras that are of special interest to us: the tensor superalgebra $T(\mathfrak{g})$ and the UESA $\mathfrak{U}(\mathfrak{g})$. $T(\mathfrak{g})$ is graded by taking $T(\mathfrak{g})_{(n)}$ as the tensors of degree n. As any graded algebra, $T(\mathfrak{g})$ is also filtered by taking $T(\mathfrak{g})^n = \bigoplus_{0 \leq i \leq n} T(\mathfrak{g})_{(i)}$.

Since the ideal $I \subset T(\mathfrak{g})$ generated by the elements $X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X, Y]$ in the tensor superalgebra $T(\mathfrak{g})$ is not homogeneous (i.e. is not generated by homogeneous elements) we cannot expect $\mathfrak{U}(\mathfrak{g})$ to be graded. However it is filtered with $\mathfrak{U}(\mathfrak{g})^n := \pi(T(\mathfrak{g})^n)$ (recall $\pi \colon T(\mathfrak{g}) \to T(\mathfrak{g})/I \cong \mathfrak{U}(\mathfrak{g})$). The next proposition clarifies the structure of $\mathfrak{U}(\mathfrak{g})$ as filtered superalgebra.

Proposition 1.6.9. *Let $X_1, \dots, X_M, X_{M+1}, \dots, X_N$ be a basis for the Lie superalgebra $\mathfrak{g}$, $X_1 \dots X_M$ even, $X_{M+1} \dots X_N$ odd. Then:*

(i) 1, $X_{k_1}, \dots, X_{k_r}, X_{k_{r+1}}, \dots, X_{k_{r+s}}$, $k_1 \leq \cdots \leq k_r < k_{r+1} < \cdots < k_{r+s}$, X_{k_i}

even for $1 \leq i \leq r$, X_{k_j} *odd for* $r+1 \leq j \leq r+s$, *is a basis for* $\mathfrak{U}(\mathfrak{g})^n$, $r+s \leq n$.

(ii) *Moreover* $\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$ *is commutative and the natural map* $\mathfrak{g} \to \mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$ *extends to an algebraic isomorphism of* $\mathrm{Sym}(\mathfrak{g})$ *onto* $\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$.

Proof. (i) By induction on $n = r + s$. The cases $n = 0, 1$ are clear. Let $n > 1$. Since $X_{j_1} \otimes \cdots \otimes X_{j_l}$, $l \leq n$, generate $T(\mathfrak{g})^n$, we have that $X_{j_1} \ldots X_{j_l}$, $l \leq n$, generate $\mathfrak{U}(\mathfrak{g})^n$. By PBW Theorem 1.6.5 the standard monomials are linearly independent, so we only need to prove they generate $\mathfrak{U}(\mathfrak{g})^n$. While proving the PBW Theorem we showed that

$$T(\mathfrak{g})^n \subset I + \sum_{1 \leq r \leq n} T_r^0,$$

hence $\mathfrak{U}(\mathfrak{g})^n \subset \sum_{r=1}^n \pi(T_r^0)$, which is what we want.

(ii) To prove $\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$ is commutative, notice that since

$$XY - (-1)^{|X||Y|} YX = [X, Y] \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g}),$$

we have

$$X_{k_1} \ldots X_{k_j} - (-1)^u X_{\sigma(k_1)} \ldots X_{\sigma(k_r)} \in \mathfrak{U}(\mathfrak{g})^{r-1}$$

for a suitable u. Hence if $a \in \mathfrak{U}(\mathfrak{g})^p$ and $b \in \mathfrak{U}(\mathfrak{g})^q$, then

$$ab \equiv ba, \quad \mathrm{mod} \quad \mathfrak{U}(\mathfrak{g})^{p+q-1},$$

and this proves commutativity.

Now we build the isomorphism $\mathrm{Sym}(\mathfrak{g}) \cong \mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$. Since $\mathfrak{U}(\mathfrak{g})^1 = k \cdot 1 \oplus \mathfrak{g}$, the natural map $\mathfrak{U}(\mathfrak{g})^1 \to \mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$ induces an injection $\mathfrak{g} \to \mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$. Since $\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$ is commutative, we have a superalgebra morphism $\phi\colon \mathrm{Sym}(\mathfrak{g}) \to \mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$. The elements $X_{k_1}, \ldots, X_{k_n}$ are linearly independent in $\mathfrak{U}(\mathfrak{g})^n$ modulo $\mathfrak{U}(\mathfrak{g})^{n-1}$ (the order on the k's is specified above), hence $\phi(X_{k_1}, \ldots, X_{k_n}) = X_{k_1} \ldots X_{k_n}$ are linearly independent and form a basis of $\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))_{(n)}$. By comparing dimensions we have that ϕ is an isomorphism. □

An important application of Proposition 1.6.9 is the construction of the symmetrizer map, when k is of characteristic 0.

Let $\mathrm{char}(k) = 0$.

Consider the natural action of the permutation group S_n on $T(\mathfrak{g})_{(n)}$:

$$s \cdot X_1 \otimes \cdots \otimes X_n := (-1)^{p(s)} X_{s^{-1}(1)} \otimes \cdots \otimes X_{s^{-1}(n)}$$

where

$$p(s) := |\{(k, l) \mid k > l,\ X_k, X_l \text{ odd},\ s(k) < s(l)\}|.$$

Notice that $p(s)$ has the following interpretation. It is the (unique) function $p\colon S_n \to \mathbb{Z}^+$ such that if $X_1, \ldots, X_n \in \mathrm{Sym}(\mathfrak{g})$:

$$X_{s^{-1}(1)} \ldots X_{s^{-1}(n)} = (-1)^{p(s)} X_1 \ldots X_n, \quad s \in S_n.$$

Definition 1.6.10. Let $t \in T(\mathfrak{g})_{(n)}$. We say that t is *symmetric* if $s \cdot t = t$, for all $s \in S_n$. We denote by $\overline{T(\mathfrak{g})}_{(n)}$ the homogeneous symmetric tensors of degree n and with $\overline{T(\mathfrak{g})} = \bigoplus_{n \geq 0} \overline{T(\mathfrak{g})}_{(n)}$.

Proposition 1.6.11. *The linear morphism* $Q_n \colon T(\mathfrak{g}) \to T(\mathfrak{g})$ *defined as*

$$Q_0 = 1, \quad Q_n(t) = \frac{1}{n!} \sum_{s \in S_n} s \cdot t$$

is a projection onto the symmetric tensors $\overline{T(\mathfrak{g})}$.

Proof. Direct check. □

Lemma 1.6.12. *Let the notation be as above and let* $\pi \colon T(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$.

(1) $\pi \colon \overline{T(\mathfrak{g})} \to \mathfrak{U}(\mathfrak{g})$ *is a linear isomorphism preserving the filtration, that is,* $\pi(\overline{T(\mathfrak{g})}^n) = \mathfrak{U}(\mathfrak{g})^n$.

(2) $T(\mathfrak{g}) = I + \overline{T(\mathfrak{g})}$.

Proof. By induction on the $\mathbb{Z}$-grading of $\overline{T(\mathfrak{g})}$. For $n = 0$ we have that π clearly induces an isomorphism of $\overline{T(\mathfrak{g})}^0$ with $\mathfrak{U}(\mathfrak{g})^0$. Let $t = X_1 \otimes \cdots \otimes X_n$, $\tau = (i, i+1) \in S_n$. We have

$$\begin{aligned} t - \tau \cdot t = {} & X_1 \otimes \dots X_i \otimes X_{i+1} \otimes \cdots \otimes X_n \\ & (-1)^{|X_i||X_{i+1}|} X_1 \otimes \cdots \otimes X_{i+1} \otimes X_i \cdots \otimes X_n \\ = {} & X_1 \otimes \cdots \otimes [X_i, X_{i+1}] \cdots \otimes X_n + I \in T(\mathfrak{g})^{n-1} + I. \end{aligned}$$

Since any $s \in S_n$ is the product of adjacent transpositions we have $u - s \cdot u \in T(\mathfrak{g})^{n-1} + I$ for all $u \in T(\mathfrak{g})^n$. Averaging over S_n we get

$$u - Q_n(u) \in T(\mathfrak{g})^{n-1} + I.$$

Applying π to this relation we get

$$\mathfrak{U}(\mathfrak{g})^n \subset \mathfrak{U}(\mathfrak{g})^{n-1} + \pi(\overline{T(\mathfrak{g})}^n).$$

By induction, comparing dimensions, we get the result.

(2) is a consequence of (1). □

Let $\bar{\mathfrak{g}}$ be an abelian Lie superalgebra with the same underlying super vector space as $\mathfrak{g}$. Clearly $\mathfrak{U}(\bar{\mathfrak{g}}) = \mathrm{Sym}(\mathfrak{g})$. Let $\bar{\pi} \colon T(\bar{\mathfrak{g}}) \to \mathfrak{U}(\bar{\mathfrak{g}})$. $J = \ker \bar{\pi}$ is the ideal in $T(\mathfrak{g})$ generated by the elements

$$X \otimes Y - (-1)^{|X||Y|} Y \otimes X.$$

Lemma 1.6.13. *Let* $\mathrm{Sym}(\mathfrak{g})_{(n)}$ *be the space of elements of degree* n *in* $\mathrm{Sym}(\mathfrak{g})$. *Consider the filtration of the symmetric superalgebra* $\mathrm{Sym}(\mathfrak{g})^n = \bigoplus_{i \leq n} \mathrm{Sym}(\mathfrak{g})_{(i)}$. *Then* $\bar{\pi}\colon \overline{T(\mathfrak{g})} \to \mathrm{Sym}(\mathfrak{g})$ *is a linear isomorphism preserving the filtration, i.e.,* $\bar{\pi}(\overline{T(\mathfrak{g})}^n) = \mathrm{Sym}(\mathfrak{g})^n$.

Proof. As in Lemma 1.6.12. □

We are ready to define the symmetrizer map.

Definition 1.6.14. We define the *symmetrizer* of $\mathrm{Sym}(\mathfrak{g})$ onto $\mathfrak{U}(\mathfrak{g})$ as the unique linear map S making the following diagram commute:

$$\begin{array}{ccc} \overline{T(\mathfrak{g})} & \xrightarrow{\ \pi\ } & \mathfrak{U}(\mathfrak{g}) \\ \downarrow{\scriptstyle \bar{\pi}} & \nearrow{\scriptstyle S} & \\ \mathrm{Sym}(\mathfrak{g}). & & \end{array}$$

Theorem 1.6.15. *The symmetrizer map* S *is a linear isomorphism and preserves the filtration; moreover*

$$S(\bar{\pi}(X_1)\dots\bar{\pi}(X_n)) = \frac{1}{n!}\sum_{s \in S_n} (-1)^{p(s)} X_{s(1)}\dots X_{s(n)}.$$

Proof. The fact that S is a linear isomorphism and preserves the filtration is clear by definition. For its expression observe that

$$\pi(Q_n(X_1 \otimes \cdots \otimes X_n)) = \frac{1}{n!}\sum_{s \in S_n} (-1)^{p(s)} X_{s(1)}\dots X_{s(n)}$$

and

$$\bar{\pi}(Q_n(X_1 \otimes \cdots \otimes X_n)) = \bar{\pi}(X_1)\dots\bar{\pi}(X_n).$$

By the commutativity of the above diagram we are done. □

In conclusion of this section, we want to remark that in [22] the authors take a radically different point of view in proving the statement of the PBW Theorem. They use the symmetrizer linear isomorphism to transfer the product structure from $\mathfrak{U}(\mathfrak{g})$ to a new product structure $*$ in $\mathrm{Sym}(\mathfrak{g})$, and then they prove that $(\mathrm{Sym}(\mathfrak{g}), *) \cong \mathfrak{U}(\mathfrak{g})$ as superalgebras. Their proof holds over a field of characteristic zero, however it is more general in the sense that it holds for a Lie algebra object in an arbitrary tensor category.

Another different and more general proof than the one that we present is found also in [76]. There, the statement is given for a Lie superalgebra over k commutative ring with 1, where 2 and 3 are invertible elements. In this case, the technique of the proof makes essential use of the “Diamond Lemma” by G. Bergman [12].

1.7 Hopf superalgebras

In this section we briefly discuss Hopf superalgebras and some examples taken from ordinary geometry. As we shall see, Hopf superalgebras are important in the understanding of Lie supergroups and affine algebraic supergroups since they represent an alternative way to approach and discuss them.

Definition 1.7.1. We say that the superalgebra A (not necessarily commutative) is a *Hopf superalgebra* if A has the following properties.

1. A is a *superalgebra*, with multiplication $\mu\colon A \otimes A \to A$ and unit $i : k \to A$.

2. A is a super *coalgebra*, that is, we can define two linear maps called *comultiplication* $\Delta\colon A \to A \otimes A$ and *counit* $\epsilon\colon A \to k$ with the following properties:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mathrm{id}\otimes\epsilon} & A\otimes k \\ \Delta\uparrow & & \uparrow\cong \\ A & \xrightarrow{\mathrm{id}} & A, \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\epsilon\otimes\mathrm{id}} & k\otimes A \\ \Delta\uparrow & & \uparrow\cong \\ A & \xrightarrow{\mathrm{id}} & A, \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\Delta\otimes\mathrm{id}} & A\otimes A\otimes A \\ \Delta\uparrow & & \uparrow\mathrm{id}\otimes\Delta \\ A & \xrightarrow{\Delta} & A. \end{array}$$

A *morphism* $\phi\colon A \to B$ of two super coalgebras with comultiplication and counit Δ_A, Δ_B and ϵ_A, ϵ_B respectively is a linear map such that $(\phi\otimes\phi)\cdot\Delta_A = \Delta_B\cdot\phi, \epsilon_B\cdot\phi = \epsilon_A$.

3. The multiplication μ and the unit i are super coalgebra morphisms.

4. The comultiplication Δ and the counit ϵ are superalgebra morphisms.

5. A is equipped with a bijective linear map $S\colon A \to A$ called the *antipode* such that the following diagrams commute:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{S\otimes\mathrm{id}} & A\otimes A \\ \Delta\uparrow & & \uparrow\mu \\ A & \xrightarrow{i\cdot\epsilon} & A, \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\mathrm{id}\otimes S} & A\otimes A \\ \Delta\uparrow & & \uparrow\mu \\ A & \xrightarrow{i\cdot\epsilon} & A. \end{array}$$

A superalgebra A satisfying the first four properties is called a super *bialgebra*.

A *Hopf superalgebra morphism* is a linear map $\phi\colon A \to B$ which is a morphism of both the superalgebra and super coalgebra structures of A and B, and in addition

$$S_B \cdot \phi = \phi \cdot S_A,$$

where S_A and S_B denote respectively the antipodes in A and B.

We say that I is a *Hopf ideal* of a Hopf superalgebra A if I is a two-sided ideal, and moreover

$$\Delta(I) \subset I \otimes A + A \otimes I, \quad \epsilon(I) = 0, \quad S(I) \subset I.$$

One can check immediately that the superalgebra A/I inherits naturally a Hopf superalgebra structure from A.

I is a *coideal* of a coalgebra A if it is an abelian subgroup of A and

$$\Delta(I) \subset I \otimes A + A \otimes I.$$

There are many interesting examples of Hopf algebras; we refer the reader to [20], [59], [72] for a comprehensive treatment.

In particular to any affine algebraic group in ordinary geometry, we can associate two very important Hopf algebras: $\mathcal{O}(G)$, the commutative Hopf algebra of algebraic functions on G, and $\mathcal{U}(\mathrm{Lie}(G))$, the universal enveloping algebra of the Lie algebra $\mathrm{Lie}(G)$. In the case of $\mathcal{O}(G)$ the comultiplication Δ and the counit ϵ are given as follows:

$$\Delta(f)(x \otimes y) := f(xy), \quad \epsilon(f) = f(1_G).$$

In the case of $\mathcal{U}(\mathrm{Lie}(G))$ the comultiplication Δ and the counit ϵ are given on the generators $\mathfrak{g}$ as follows:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \epsilon(X) = 0 \quad \text{for all } X \in \mathfrak{g}.$$

Under suitable conditions, these two Hopf algebras are in duality with each other, as is explained in [20], Ch. 7.

1.8 The even rules

We conclude our treatment of linear supergeometry by discussing the *even rules*.

The even rules are a device that enables us to work with purely even objects and has a variety of applications in supergeometry. In order to use them, however, we have to show the functoriality of the construction we are interested in, under *extension of scalars*. We shall not take advantage systematically of the even rules, to go back and forth between super (i.e., $\mathbb{Z}_2$-graded) and purely even objects; however, since the even rules are often employed in the physics literature, sometimes implicitly, we shall briefly outline the key ideas behind them. For a more detailed account we invite the reader to look at [22], p. 57.

Definition 1.8.1. Let V a super vector space and let A be a commutative superalgebra. We define

$$V_A := A \otimes V, \quad V(A) := (A \otimes V)_0,$$

where V_A is an A-module, while $V(A)$, the even part of V_A, is an A_0-module. We also say that V_A is obtained by $V(A)$ by *extension of scalars*.

It is clear that if we have a linear morphism $f: V_1 \otimes V_2 \to V$, such a morphism induces an A_0-linear map $f(A): V_1(A) \otimes V_2(A) \to V(A)$, (or if we prefer an A_0-multilinear map $(f_1(A), f_2(A)): V_1(A) \times V_2(A) \to V(A)$) which is functorial in A. The next proposition establishes a converse for this fact.

Proposition 1.8.2. *Any A_0-multilinear map $(f_1(A), f_2(A)) \colon V_1(A) \times V_2(A) \to V(A)$ functorial in A comes from a unique linear morphism $f : V_1 \otimes V_2 \to V$.*

Proof (Sketch). We start by proving uniqueness. We want to determine $f(v_1 \otimes v_2)$ by the knowledge of the multilinear morphism $f(A) = (f_1(A), f_2(A))$, for $A = k[\theta_1, \theta_2]$. Define the element in $V(A)$:

$$w_i = \begin{cases} v_i & \text{if } v_i \text{ even}, \\ \theta_i v_i & \text{if } v_i \text{ odd}. \end{cases}$$

For concreteness assume that v_1 even and v_2 odd. Then we have

$$f(A)(w_1, w_2) = f(A)(v_1, \theta_2 v_2) = \theta_2 f(v_1 \otimes v_2).$$

This formula determines uniquely $f(v_1 \otimes v_2)$. This formula moreover also gives us existence since it provides a way to define $f(v_1 \otimes v_2)$. □

Remark 1.8.3. (1) There is an obvious generalization of the previous result to the case of linear morphisms $f \colon \otimes_{i \in I} V_i \to V$. The reasoning is similar, and we invite the reader to consult [22] for more details.

(2) The previous result holds if A is just taken to be an exterior algebra, $A = k[\xi_1 \dots \xi_n]$, in other words, we can substantially weaken our hypothesis. This is clear by looking at the proof.

(3) The same result holds also for modules over some commutative superalgebra R replacing our field k while A runs over commutative R-superalgebras and $V(A)$ is the A_0-module $(A \otimes_R V)_0$ as before. The proof goes practically unchanged.

Proposition 1.8.2 has the following very useful corollary, which allows us to extend in a straightforward manner algebraic structures from $V(A)$ to V_A.

Corollary 1.8.4. *Let the notation be as above. In order to give a superalgebra structure on a super vector space V, it is enough to give a functorial structure of A_0-algebra on $V(A)$. The superalgebra will be Lie, respectively associative or commutative and so on, if and only if the $V(A)$ A_0-algebras are.*

Proof. The first statement is clear since a superalgebra structure comes precisely from the morphisms considered in Proposition 1.8.2. As for the Lie, associative and commutative properties, they rely on commutative diagrams constructed starting from such morphisms. Proposition 1.8.2 ensures the commutativity of such diagrams by the functoriality of the corresponding diagrams for $V(A)$. We leave to the reader all the checks involved in this construction. □

1.9 References

All of the material appearing here is standard and can be found in one of the references [56], [76], [10], [49].

2

Sheaves, functors and the geometric point of view

Classically, sheaf theory provides an alternative and very elegant way to look at theories, like differentiable manifolds and algebraic varieties, which originally were introduced using very different methods. On the other hand, in supergeometry, its use is unavoidable since at the very start we need sheaves in order to define properly any supergeometric object in differential or algebraic category.

We start our treatment by discussing sheaves of functions in the familiar setting of differentiable manifolds and algebraic varieties and then we go on to the more abstract definition of sheaves in general, arriving finally at ringed spaces and locally ringed spaces, which is our starting point for the development of supergeometry, as we shall see in the next chapter.

We also give a brief introduction to schemes and their functor of points, discussing the examples of projective space and Grassmannian variety. The language of the functor of points and its generalization to supergeometry are extremely important to us since they provide geometric intuition to an otherwise very abstract setting. In the end, we recall some results on coherent sheaves, that we shall need in the sequel.

We do not present here proofs for most of our statements, which are all classical and well known; a treatment of sheaf theory and ringed spaces can be found. for example. in [43], Ch. II, or in [29], Ch. I, while a comprehensive functorial treatment of algebraic geometry, via the functor of points, is found in [23].

2.1 Ringed spaces of functions

A ringed space is a broad concept in which we can fit most of the interesting geometrical objects. It consists of a topological space together with a sheaf of functions on it. Let us examine some familiar and important examples.

Let M be a differentiable manifold, whose topological space is Hausdorff and second countable. For each open set $U \subset M$, let $C^\infty(U)$ be the $\mathbb{R}$-algebra of smooth functions on U.

The assignment

$$U \mapsto C^\infty(U)$$

satisfies the following two properties:

(1) If $U \subset V$ are two open sets in M, we can define the *restriction* map, which is an algebra morphism:

$$\mathrm{r}_{V,U} : C^\infty(V) \to C^\infty(U), \quad f \mapsto f|_U,$$

which is such that

(i) $\mathrm{r}_{U,U} = \mathrm{id}$,

(ii) $\mathrm{r}_{W,U} = \mathrm{r}_{V,U} \circ \mathrm{r}_{W,V}$.

(2) Let $\{U_i\}_{i \in I}$ be an open covering of U and let $\{f_i\}_{i \in I}$, $f_i \in C^\infty(U_i)$, be a family such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In other words the elements of the family $\{f_i\}_{i \in I}$ agree on the intersection of any two open sets $U_i \cap U_j$. Then there exists a unique $f \in C^\infty(U)$ such that $f|_{U_i} = f_i$.

We leave to the reader, as an exercise, to check that indeed the assignment satisfies the two given properties. As for property (1), the existence of the restriction map satisfying equations (i) and (ii) is clear while for the property (2), also-called the *gluing property*, it is a simple exercise.

Such an assignment is called a *sheaf*. The pair (M, C^∞), consisting of the topological space M, underlying the differentiable manifold, and the sheaf of the C^∞ functions on M is an example of *locally ringed space* (the word "locally" refers to a local property of the sheaf of C^∞ functions we shall discuss in the next sections). We shall examine in detail locally ringed spaces and their properties in the next section.

Remark 2.1.1. It is customary to denote a differentiable manifold and its underlying topological space with the same symbol. In this section, we follow this convention, however starting from the next section, we shall mark the difference between the manifold M and its underlying topological space $|M|$. We shall also do the same for algebraic varieties.

Given two manifolds M and N, and the respective sheaves of smooth functions C^∞_M and C^∞_N, a morphism f from M to N, viewed as ringed spaces, is a morphism $|f|\colon M \to N$ of the underlying topological spaces together with a morphism of algebras,

$$f^*\colon C^\infty_N(V) \to C^\infty_M(f^{-1}(V)), \quad f^*(\phi)(x) = \phi(|f|(x)), \quad V \text{ open in } N,$$

compatible with the restriction morphisms.

Notice that, as soon as we give the continuous map $|f|$ between the topological spaces, the morphism f^* is automatically assigned. This is a peculiarity of the sheaf of smooth functions on a manifold. Such a property is no longer true for a generic ringed space and, in particular, as we shall see, it is not true for supermanifolds.

A morphism of differentiable manifolds gives rise to a unique (locally) ringed space morphism and vice versa.

Moreover, given two manifolds, they are isomorphic as manifolds if and only if they are isomorphic as (locally) ringed spaces. In the language of categories, we say we have a fully faithful functor from the category of manifolds to the category of locally ringed spaces.

Before going to the general treatment, let us consider another interesting example arising from classical algebraic geometry. As we shall see in the next chapters, the generalization of algebraic geometry to the super-setting comes somehow more naturally than the similar generalization of differentiable geometry. This is because the machinery of algebraic geometry was developed to take already into account the presence of (even) nilpotents and consequently, the language is more suitable to supergeometry.

Let X be an affine algebraic variety in the affine space $\mathbb{A}^n$ over an algebraically closed field k and let $\mathcal{O}(X) = k[x_1, \dots, x_n]/I$ be its coordinate ring, where the ideal I is prime. This corresponds topologically to the irreducibility of the variety X. We can think of the points of X as the zeros of the polynomials in the ideal I in $\mathbb{A}^n$. X is a topological space with respect to the Zariski topology, whose closed sets are the zeros of the polynomials in the ideals of $\mathcal{O}(X)$ (see [43], Ch. I, for a complete discussion of the Zariski topology). For each open U in X, consider the assignment

$$U \mapsto \mathcal{O}_X(U),$$

where $\mathcal{O}_X(U)$ is the k-algebra of regular functions on U. By definition, these are the functions $f : X \to k$ that can be expressed as a quotient of two polynomials at each point of $U \subset X$.

As in the case of differentiable manifolds, our assignment satisfies the properties (1) and (2) described above. The first property is clear while for the second one we refer the reader to [43], Ch. II, Section 2.

The assignment $U \mapsto \mathcal{O}_X(U)$ is another example of a sheaf and we shall call it the *structure sheaf* of the variety X or the sheaf of regular functions. $(X, \mathcal{O}_X)$ is another example of a (locally) ringed space.

2.2 Sheaves and ringed spaces

We are now going to formulate more generally the notion of sheaf and of ringed space that we have described in two specific examples of the previous section.

Definition 2.2.1. Let $|M|$ be a topological space. A *presheaf* of commutative algebras $\mathcal{F}$ on X is an assignment

$$U \mapsto \mathcal{F}(U), \quad U \text{ open in } |M|, \quad \mathcal{F}(U) \text{ a commutative algebra},$$

such that the following holds:

(1) If $U \subset V$ are two open sets in $|M|$, there exists a morphism $\mathrm{r}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the *restriction morphism* and often denoted by $\mathrm{r}_{V,U}(f) = f|_U$, such that

(i) $\mathrm{r}_{U,U} = \mathrm{id}$,

(ii) $\mathrm{r}_{W,U} = \mathrm{r}_{V,U} \circ \mathrm{r}_{W,V}$.

A presheaf $\mathcal{F}$ is called a *sheaf* if the following holds:

(2) Given an open covering $\{U_i\}_{i \in I}$ of U and a family $\{f_i\}_{i \in I}$, $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$.

The elements in $\mathcal{F}(U)$ are called *sections* over U; when $U = |M|$ we call such elements *global sections*.

The assignments $U \mapsto C^\infty(U)$, U open in the differentiable manifold M and $U \mapsto \mathcal{O}_X(U)$, U open in the algebraic variety X, described in the previous section, are examples of sheaves of functions on the topological spaces $|M|$ and $|X|$ underlying the differentiable manifold M and the algebraic variety X respectively.

Observation 2.2.2. In the language of categories, property (1) in Definition 2.2.1 says that we have defined a functor, $\mathcal{F}$, from top(M) to (alg), where top(M) is the category of the open sets in the topological space $|M|$, the arrows given by the inclusions of open sets while (alg) is the category of commutative algebras. In fact, the assignment $U \mapsto \mathcal{F}(U)$ defines $\mathcal{F}$ on the objects while the assignment

$$U \subset V \mapsto \mathrm{r}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

defines $\mathcal{F}$ on the arrows.

Hence we can give the following equivalent definition of presheaf and sheaf of commutative algebras.

Definition 2.2.3 (Alternative definition of presheaf and sheaf). Let $|M|$ be a topological space. We define a *presheaf of algebras* on $|M|$ to be a functor

$$\mathcal{F} : \ \mathrm{top}(M)^{\mathrm{op}} \to (\mathrm{alg}).$$

The suffix "op" denotes as usual the opposite category; in other words, $\mathcal{F}$ is a contravariant functor from top(M) to (alg). A presheaf is a *sheaf* if it satisfies the property (2) of Definition 2.2.1.

If $\mathcal{F}$ is a (pre)sheaf on $|M|$ and U is open in $|M|$ we define $\mathcal{F}|_U$, *the (pre)sheaf* $\mathcal{F}$ *restricted to* U, as the functor $\mathcal{F}$ restricted to the category of open sets in U (viewed as a topological space itself). In other words, we restrict our attention to the assignment $V \mapsto \mathcal{F}(V)$ for just the open sets V in U.

We have defined sheaves as functors with values in the category of commutative algebras, however, in the same way, we can also define sheaves with values in commutative rings, groups, sets or other algebraic structures and their super correspondents. Of course, depending on the structure of the category of arrival, the restriction morphisms have to be taken as the appropriate morphisms.

From now on, we shall speak only of "presheaf" and of "sheaf", without further specifications, whenever our statements are independent of the category of arrival.

A most important object associated to a given presheaf is the *stalk* at a point.

Definition 2.2.4. Let $\mathcal{F}$ be a presheaf on the topological space $|M|$ and let x be a point in $|M|$. We define the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$, at the point x, as the direct limit

$$\varinjlim \mathcal{F}(U),$$

where the direct limit is taken for all the U open neighbourhoods of x in $|M|$. $\mathcal{F}_x$ consists of the disjoint union of all pairs (U, s) with U open in $|M|$, $x \in U$, and $s \in \mathcal{F}(U)$, modulo the equivalence relation: $(U, s) \cong (V, t)$ if and only if there exists a neighbourhood W of x, $W \subset U \cap V$, such that $s|_W = t|_W$.

The elements in $\mathcal{F}_x$ are called *germs of sections*.

Definition 2.2.5. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $|M|$. A *morphism of presheaves* $\phi\colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, for each open set U in $|M|$, such that for all $V \subset U$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ {\scriptstyle r_{U,V}}\downarrow & & \downarrow{\scriptstyle r_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V). \end{array}$$

Equivalently and more elegantly, one can also say that a morphism of presheaves is a natural transformation between the two presheaves $\mathcal{F}$ and $\mathcal{G}$ viewed as functors (see the alternative definition of sheaf in Definition 2.2.3).

A *morphism of sheaves* is just a morphism of the underlying presheaves.

Clearly any morphism of presheaves induces a morphism on the stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. The sheaf property, i.e., property (2) in Definition 2.2.1, ensures that if we have two morphisms of sheaves ϕ and ψ such that $\phi_x = \psi_x$ for all x, then $\phi = \psi$.

We say that the morphism of sheaves ϕ is *injective* (*resp. surjective*) if ϕ_x is injective (resp. surjective).

On the notion of surjectivity, however, one should exert some care, since we can have a surjective sheaf morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is *not* surjective for some open sets U. This strange phenomenon is a consequence of the following fact. While the assignment $U \mapsto \ker(\phi(U))$ always defines a sheaf, the assignment

$$U \mapsto \operatorname{im}(\phi(U)) = \mathcal{F}(U)/\mathcal{G}(U)$$

defines in general only a presheaf and not all the presheaves are sheaves. A simple example is given by the assignment associating to an open set U in $\mathbb{R}$ the algebra of constant real functions on U. Clearly this is a presheaf, but not a sheaf.

We can always associate, in a natural way, to any presheaf a sheaf called its *sheafification*. Intuitively, one may think of the sheafification as the sheaf that best "approximates" the given presheaf. For example, the sheafification of the presheaf of constant functions on open sets in $\mathbb{R}$ is the sheaf of locally constant functions on open sets in $\mathbb{R}$.

We construct the sheafification of a presheaf using the *étalé space*, which we also need in the sequel, since it gives an equivalent approach to sheaf theory.

Definition 2.2.6. Let $\mathcal{F}$ be a presheaf on $|M|$. We define the *étalé space* of $\mathcal{F}$ to be the disjoint union $\coprod_{x \in |M|} \mathcal{F}_x$. For each open $U \subset |M|$ and each $s \in \mathcal{F}(U)$ define the map $\hat{s}_U : U \to \coprod_{x\in|U|} \mathcal{F}_x$, $\hat{s}_U(x) = s_x$. We give to the étalé space the finest topology that makes the maps $\hat{s}_U$ continuous, for all open $U \subset |M|$ and all sections $s \in \mathcal{F}(U)$. We define $\mathcal{F}_{\text{et}}$ to be the presheaf on $|M|$:

$$U \mapsto \mathcal{F}_{\text{et}}(U) = \{\hat{s}_U : U \to \coprod_{x\in|U|} \mathcal{F}_x,\ \hat{s}_U(x) = s_x \in \mathcal{F}_x\}.$$

As we shall see, the presheaf $\mathcal{F}_{\text{et}}$ is actually a sheaf and it provides an explicit construction of the sheafification of the presheaf $\mathcal{F}$.

Definition 2.2.7. Let $\mathcal{F}$ be a presheaf on $|M|$. A *sheafification* of $\mathcal{F}$ is a sheaf $\tilde{\mathcal{F}}$, together with a presheaf morphism $\alpha\colon \mathcal{F} \to \tilde{\mathcal{F}}$, such that

(1) any presheaf morphism $\psi\colon \mathcal{F} \to \mathcal{G}$, $\mathcal{G}$ a sheaf, factors via α, i.e., $\psi\colon \mathcal{F} \xrightarrow{\alpha} \tilde{\mathcal{F}} \to \mathcal{G}$,

(2) $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are locally isomorphic, i.e., there exists an open cover $\{U_i\}_{i\in I}$ of $|M|$ such that $\mathcal{F}(U_i) \cong \tilde{\mathcal{F}}(U_i)$ via α.

The next proposition establishes the existence of the sheafification of a presheaf.

Proposition 2.2.8. *Let $\mathcal{F}$ be a presheaf on $|M|$.*

(1) *The sheafification $\tilde{\mathcal{F}}$ of $\mathcal{F}$ always exists and it is unique.*

(2) *The presheaf $\mathcal{F}_{\text{et}}$ is a sheaf and it is the sheafification of the presheaf $\mathcal{F}$ with $\alpha\colon \mathcal{F} \to \mathcal{F}_{\text{et}}$ defined as $\alpha_U(s)(x) = s_x$, $s \in \mathcal{F}(U)$, $x \in U$.*

In particular $\mathcal{F}$ is a sheaf if and only if $\mathcal{F} \cong \tilde{\mathcal{F}} = \mathcal{F}_{\text{et}}$.

Proof. See [43], p. 64. □

The previous discussion enables us to properly define the quotient of sheaves.

Definition 2.2.9. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of rings on some topological space $|M|$. Assume that we have an injective morphism of sheaves $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for all U open in $|M|$. We define the *quotient* $\mathcal{F}/\mathcal{G}$ to be the sheafification of the image presheaf: $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$.

In general, $(\mathcal{F}/\mathcal{G})(U) \neq \mathcal{F}(U)/\mathcal{G}(U)$, however they are locally isomorphic by Definition 2.2.7.

There is a most effective way to define a sheaf and a sheaf morphism, which proves to be very useful for applications. We just state the results, referring to [29], Ch. I, for more details.

Definition 2.2.10. Let $\mathcal{B}$ be a base for the open sets in the topological space $|M|$. The assignment $U \mapsto \mathcal{F}(U)$, for all $U \in \mathcal{B}$, is called a $\mathcal{B}$-*sheaf* if it satisfies the condition (1) as in Definition 2.2.1 for the open sets in $\mathcal{B}$ and the following modification of the condition (2). For all open U in $\mathcal{B}$, let $\{U_i\}_{i\in I}$ be a covering in $\mathcal{B}$ of U and $\{f_i\}_{i\in I}$ a family such that $f_i \in \mathcal{F}(U_i)$. If $f_i|_V = f_j|_V$ for all $V \subset U_i \cap U_j$, $V \in \mathcal{B}$, then there exists a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$.

Proposition 2.2.11. *Let $\mathcal{B}$ be a base for the open sets in the topological space $|M|$.*

(1) *Every $\mathcal{B}$-sheaf extends uniquely to a sheaf on $|M|$.*

(2) *If $\mathcal{G}$ and $\mathcal{H}$ are two sheaves on $|M|$ and for all U in $\mathcal{B}$ we have a collection of morphisms*

$$\psi_U : \mathcal{G}(U) \to \mathcal{H}(U)$$

commuting with restrictions, then there is a unique sheaf morphism $\phi : \mathcal{G} \to \mathcal{H}$ such that $\phi_U = \psi_U$ for all $U \in \mathcal{B}$.

Proposition 2.2.12. *Let $\{U_i\}_{i \in I}$ be an open covering of the topological space $|M|$. Assume the following:*

(1) *We have defined sheaves $\mathcal{F}_{U_i}$ for all i.*

(2) *$\phi_{U_i U_j} : \mathcal{F}_{U_i}|_{U_i \cap U_j} \to \mathcal{F}_{U_j}|_{U_i \cap U_j}$ are isomorphisms satisfying the compatibility conditions*

$$\phi_{U_i U_j} \phi_{U_k U_i} = \phi_{U_k U_j} \quad \text{on } U_i \cap U_j \cap U_k \text{ for all } i, j, k \in I.$$

Then there exists a unique sheaf $\mathcal{F}$ on $|M|$ such that $\mathcal{F}|_U = \mathcal{F}_{U_i}$.

We are ready to define ringed spaces.

Definition 2.2.13. We define *ringed space* to be a pair $M = (|M|, \mathcal{F})$ consisting of a topological space $|M|$ and a sheaf of commutative rings $\mathcal{F}$ on $|M|$. We say that the ringed space $(|M|, \mathcal{F})$ is a *locally ringed space*, if the stalk $\mathcal{F}_x$ is a local ring for all $x \in |M|$.

A *morphism* of ringed spaces $\phi : M = (|M|, \mathcal{F}) \to N = (|N|, \mathcal{G})$ consists of a morphism $|\phi| : |M| \to |N|$ of the topological spaces (in other words, $|\phi|$ is a continuous map) and a sheaf morphism $\phi^* : \mathcal{O}_N \to \phi_* \mathcal{O}_M$ where $\phi_* \mathcal{O}_M$ is the sheaf on $|N|$ defined as follows: $(\phi_* \mathcal{O}_M)(U) = \mathcal{O}_M(\phi^{-1}(U))$ for all U open in $|N|$. A morphism of ringed spaces induces a morphism on the stalks for each $x \in |M|$: $\phi_x : \mathcal{O}_{N,|\phi|(x)} \to \mathcal{O}_{M,x}$. If M and N are locally ringed spaces, we say that the morphism of ringed spaces ϕ is a *morphism of locally ringed spaces* if ϕ_x is local, i.e., $\phi_x^{-1}(m_{M,x}) = m_{N,|\phi|(x)}$ where $m_{N,|\phi|(x)}$ and $m_{M,x}$ are the maximal ideals in the local rings $\mathcal{O}_{N,|\phi|(x)}$ and $\mathcal{O}_{M,x}$, respectively.

Observation 2.2.14. In the previous section we have seen differentiable manifolds and algebraic varieties as examples of ringed spaces. Actually both are also examples of locally ringed spaces, as one can readily verify. Moreover, one can also check that their morphisms, in the differential or in the algebraic setting respectively, correspond precisely to morphisms as locally ringed spaces.

At this point it is not hard to convince ourselves that we can take a different point of view on the definition of differentiable manifold. Namely we can equivalently define a differentiable manifold as a locally ringed space $M = (|M|, \mathcal{O}_M)$ as follows.

Definition 2.2.15. *Alternative definition of differentiable manifold.* Let M be a topological space, Hausdorff and second countable, and let $\mathcal{O}_M$ be a sheaf of commutative algebras on M, so that $(M, \mathcal{O}_M)$ is a locally ringed space. We say that $(M, \mathcal{O}_M)$ is a *real differentiable manifold* if it is locally isomorphic to the locally ringed space $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$, where $C^\infty_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$.

In the same way we can define analytic real or complex manifolds as locally ringed spaces locally isomorphic to $(\mathbb{R}^n, \mathcal{H}_{\mathbb{R}^n})$ or $(\mathbb{C}^n, \mathcal{H}_{\mathbb{C}^n})$, where $\mathcal{H}_{\mathbb{R}^n}$ and $\mathcal{H}_{\mathbb{C}^n}$ denote the sheaves of analytic functions on $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively (we leave to the reader as an exercise their definition, see [37] for more details).

It is important to keep this point of view in mind since this is the path we are going to take to define *supermanifolds*.

At this point, it is clear that we could also give an equivalent definition of algebraic variety in order to fit it into the framework of locally ringed spaces. However, as we shall see in the next section, we prefer to give a far reaching generalization of this picture, namely the notion of scheme, which turns out to be the best for our supergeometric applications.

2.3 Schemes

The concept of scheme is a step towards a further abstraction. We shall start by defining affine schemes and then we proceed to the definition of schemes in general. Our treatment is necessarily very short, for all the details and the complete story, we refer the reader to [29], Ch. I, and [43], Ch. II.

Let us start by associating to any commutative ring A its *spectrum*, that is the topological space $\mathrm{Spec}\, A$. As a set, $\mathrm{Spec}\, A$ consists of all the prime ideals in A. For each subset $S \subset A$ we define as *closed sets* in $\mathrm{Spec}\, A$:

$$V(S) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid S \subset \mathfrak{p}\} \subset \mathrm{Spec}\, A.$$

One can check that this actually defines a topology on $\mathrm{Spec}\, A$ called the *Zariski topology*.

If X is an affine variety, defined over an algebraically closed field, and $\mathcal{O}(X)$ is its coordinate ring, we have that the points of the topological space underlying X are in one-to-one correspondence with the maximal ideals in $\mathcal{O}(X)$. So we notice immediately that $\mathrm{Spec}\, \mathcal{O}(X)$ contains far more than just the points of the topological space of X; in fact it contains also all the subvarieties of X, whose information is encoded by the prime ideals in $\mathcal{O}(X)$. This tells us that the notion of scheme, we are about to introduce, is not just a generalization of the concept of algebraic variety, but it is something deeper, containing more information about the geometric objects we are interested in.

We also define the *basic open sets* in Spec A as

$$U_f := \text{Spec } A \setminus V(f) = \text{Spec } A_f \quad \text{with } f \in A,$$

where $A_f = A[f^{-1}]$ is the localization of A obtained by inverting the element f. The collection of the basic open sets U_f, for all $f \in A$, forms a base for the Zariski topology.

Next, we define the *structure sheaf* $\mathcal{O}_A$ on the topological space Spec A. In order to do this, it is enough to give an assignment $U \mapsto \mathcal{O}_A(U)$ for each basic open set $U = U_f$ in Spec A, by Proposition 2.2.11.

Proposition 2.3.1. *Let the notation be as above. The assignment*

$$U_f \mapsto A_f$$

defines a $\mathcal{B}$-sheaf on the topological space Spec A *and it extends uniquely to a sheaf of commutative rings on* Spec A*, called the* structure sheaf *and denoted by $\mathcal{O}_A$. Moreover the stalk at a point $\mathfrak{p} \in$* Spec A*, $\mathcal{O}_{A,\mathfrak{p}}$ is the localization $A_\mathfrak{p}$ of the ring A at the prime $\mathfrak{p}$.*

Proof. Direct check. (See also [29], I-18.) □

Hence, given a commutative ring A, Proposition 2.3.1 tells us that the pair $(\text{Spec } A, \mathcal{O}_A)$ is a locally ringed space that we call $\underline{\text{Spec}}\, A$, the *spectrum of the ring* A. By an abuse of notation we shall use the word *spectrum* to mean both the topological space Spec A and the locally ringed space $\underline{\text{Spec}}\, A$, the context making clear which one we are talking about.

We are finally ready for a definition of scheme. While the differentiable manifolds are locally modelled, as ringed spaces, by $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$, the schemes are geometric objects modelled by the spectrums of commutative rings.

Definition 2.3.2. We define an *affine scheme* to be a locally ringed space isomorphic to $\underline{\text{Spec}}\, A$ for some commutative ring A. We say that X is a *scheme* if $X = (|X|, \mathcal{O}_X)$ is a locally ringed space, which is locally isomorphic to affine schemes. In other words, for each $x \in |X|$, there exists an open set $U_x \subset |X|$ such that $(U_x, \mathcal{O}_X|_{U_x})$ is an affine scheme. A *morphism* of schemes is just a morphism of locally ringed spaces.

Observation 2.3.3. (1) There is an equivalence of categories between the category of affine schemes (aschemes) and the category of commutative rings (rings). This equivalence is defined on the objects by

$$(\text{rings})^{\text{op}} \to (\text{aschemes}), \quad A \mapsto \underline{\text{Spec}}\, A.$$

In particular a morphism of commutative rings $A \to B$ corresponds contravariantly to a morphism $\underline{\text{Spec}}\, B \to \underline{\text{Spec}}\, A$ of the corresponding affine superschemes. For more details see [43], Ch. II, Proposition 2.3, and [29], Ch. I, Theorem I-40. We are going to discuss in detail a generalization of this result in Chapter 10.

(2) Since any affine variety X is completely described by the knowledge of its coordinate ring $\mathcal{O}(X)$, (the ring of regular functions on the whole variety), we can associate uniquely to an affine variety X the affine scheme $\underline{\mathrm{Spec}}\,\mathcal{O}(X)$. As we noted previously, the two notions of X as algebraic variety or as a scheme are different, however they describe the same geometrical object from two different points of view. Similarly to any algebraic variety (not necessarily affine) we can associate uniquely a scheme. Moreover a morphism between algebraic varieties determines uniquely a morphism between the corresponding schemes. In the language of categories, we say we have a fully faithful functor from the category of algebraic varieties to the category of schemes. For more details see [43], Ch. II, Proposition 2.3, and [29], Ch. I. We shall show this proposition in Chapter 10 in the more general setting of supergeometry.

In the next example we describe the simplest example of a non-affine scheme: the *projective space.*

Example 2.3.4 ($\underline{\mathrm{Proj}}\,S$). Let S be a graded commutative k-algebra, k a field, i.e., $S = \bigoplus_{i\geq 0} S_i$, $S_i S_j \subset S_{i+j}$, $S_0 = k$. The elements in S_i are called *homogeneous elements of degree i*. We define $\mathrm{Proj}\,S$ as the set of all relevant homogeneous prime ideals (i.e., prime ideals generated by homogeneous elements, $\mathfrak{p} \neq \bigoplus_{i>0} S_i$). $\mathrm{Proj}\,S$ is a topological space with the closed sets defined as

$$V(I) = \{\mathfrak{p} \in \mathrm{Proj}\,S \mid \mathfrak{p} \supset I\}.$$

Define W_f as the open set

$$W_f = |\,\mathrm{Proj}\,S| - V(f),$$

where (f) is the ideal generated by the homogeneous element f. The points of W_f can be identified with the homogeneous primes of $S[f^{-1}]$, which in turn correspond to the primes in the ring of the elements of degree zero in $S[f^{-1}]$, denoted by $S[f^{-1}]_0$. Hence we can identify W_f with the topological space $\mathrm{Spec}\,S[f^{-1}]_0$, which has an affine scheme structure by its very definition. If $x_0, \dots, x_n$ are elements of degree 1 that generate the ideal $\bigoplus_{i>0} S_i$, then the open sets

$$(\mathrm{Proj}\,S)_i := \mathrm{Proj}\,S - V(x_i)$$

cover $\mathrm{Proj}\,S$.

It is not hard to see, using Proposition 2.2.12, that the structure sheaves on each $(\mathrm{Proj}\,S)_i$ extend uniquely and compatibly to a sheaf on $\mathrm{Proj}\,S$ to give a locally ringed space that we denote by $\underline{\mathrm{Proj}}\,S$. By its very construction the locally ringed space $\underline{\mathrm{Proj}}\,S$ is locally isomorphic to the affine schemes $\underline{\mathrm{Spec}}\,S[x_i^{-1}]_0$, hence it is a scheme.

If $S = k[x_0, \dots, x_n]$ is the graded polynomial ring, we also write $\mathbb{P}^n$ for the scheme $\underline{\mathrm{Proj}}\,S$ and we call it the *projective space of dimension n*. (For more details on this construction see [29], p. 97.)

2.4 Functor of points

When we are dealing with classical manifolds and algebraic varieties, we can altogether avoid the use of their functor of points. In fact, both differentiable manifolds and algebraic varieties are well understood just by looking at their underlying topological spaces and the regular functions on the open sets.

However, if we go to the generality of schemes, the extra structure overshadows the topological points and leaves out crucial details so that we have little information, without the full knowledge of the sheaf. We shall see that the same happens for supergeometric objects. For example the evaluation of odd functions on topological points is always zero. This implies that the structure sheaf of a supermanifold cannot be reconstructed from its underlying topological space. For now let us continue our treatment of ordinary geometry.

The functor of points is a categorical device to bring back our attention to the points of a scheme; however the notion of *point* needs to be suitably generalized to go beyond the points of the topological space underlying the scheme.

Grothendieck's idea behind the definition of the functor of points associated to a scheme is the following. If X is a scheme, for each commutative ring A, we can define the set of the A-points of X in analogy to the way the classical geometers used to define the rational or integral points on a variety. The crucial difference is that we do not focus on just one commutative ring A, but we consider the A-points for all commutative rings A. In fact, the scheme we start from is completely recaptured only by the collection of the A-points for *every* commutative ring A, together with the admissible morphisms.

Let (rings) denote the category of commutative rings and (schemes) the category of schemes.

Definition 2.4.1. Let $(|X|, \mathcal{O}_X)$ be a scheme and let $T \in$ (schemes). We call the *T-points* of X, the set of all scheme morphisms $\{T \to X\}$, that we denote by $\mathrm{Hom}(T, X)$. We define the *functor of points* h_X of the scheme X as the representable functor defined on the objects as

$$h_X\colon (\text{schemes})^{\text{op}} \to (\text{sets}), \quad h_X(T) = \mathrm{Hom}(T, X).$$

Hence $h_X(T)$ are the T-points of the scheme X. The restriction of h_X to affine schemes is not in general representable. However, since, as we noticed in Observation 2.3.3, the category of affine schemes is equivalent to the category of commutative rings we have that such restriction gives a new functor h_X^a:

$$h_X^a\colon (\text{rings}) \to (\text{sets}), \quad h_X^a(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, X) = A\text{-points of } X.$$

Notice that when X is affine, $X \cong \underline{\mathrm{Spec}}\, \mathcal{O}(X)$ and we have

$$h_X^a(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, \mathcal{O}(X)) = \mathrm{Hom}(\mathcal{O}(X), A).$$

In this case the functor h_X^a is again representable.

This is a consequence of the following proposition, which comes from Observation 2.3.3.

Proposition 2.4.2. *Consider the affine schemes $X = \underline{\text{Spec}}\, \mathcal{O}(X)$ and $Y = \underline{\text{Spec}}\, \mathcal{O}(Y)$. There is a one-to-one correspondence between the scheme morphisms $X \to Y$ and the ring morphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$.*

Both h_X and h_X^a are defined on morphisms in the natural way. If $\phi\colon T \to S$ is a morphism and $f \in \text{Hom}(S, X)$ we define $h_X(\phi)(f) = f \circ \phi$. Similarly if $\psi\colon A \to B$ is a ring morphism and $g \in \text{Hom}(\mathcal{O}(X), A)$ we define $h_X^a(\psi)(g) = \psi \circ g$.

The next proposition tells us that the functors h_X and h_X^a, for a given scheme X, are not really different, but carry the same information.

Proposition 2.4.3. *The functor of points h_X of a scheme X is completely determined by its restriction to the category of affine schemes or equivalently by the functor*

$$h_X^a\colon (\text{rings}) \to (\text{sets}), \quad h_X^a(A) = \text{Hom}(\underline{\text{Spec}}\, A, X).$$

Proof. See [29], Ch. VI. □

Example 2.4.4 (The affine space). Let $\mathbb{A}^n$ be the affine space over a field k; its coordinate ring is $k[x_1, \dots, x_n]$, the ring of polynomials. Its functor of points is by definition $h_{\mathbb{A}^n}^a\colon (\text{rings}) \to (\text{sets})$, $h_{\mathbb{A}^n}^a(A) = \text{Hom}(\underline{\text{Spec}}\, A, \mathbb{A}^n)$. Since a morphism of two affine varieties corresponds contravariantly to a morphism of their coordinate rings (see Observation 2.3.3), we have $\text{Hom}(\underline{\text{Spec}}\, A, \mathbb{A}^n) = \text{Hom}(k[x_1, \dots, x_n], A)$. Any morphism $\phi\colon k[x_1, \dots, x_n] \to A$ is determined by the knowledge of $\phi(x_1) = a_1, \dots, \phi(x_n) = a_n$, $a_i \in A$. The choice of such morphism ϕ corresponds to the choice of an n-uple $(a_1, \dots, a_n)$, $a_i \in A$. So, we can identify $h_{\mathbb{A}^n}(A)$ with the set of n-uples $(a_1, \dots, a_n)$ with entries in A. If $A = \mathbb{Z}$ or $A = \mathbb{Q}$, this is the notion we already encounter in classical algebraic geometry.

The functor of points, originally introduced as a tool in algebraic geometry, can actually be employed in a much wider context.

Definition 2.4.5. Let $M = (|M|, \mathcal{O}_M)$ be a locally ringed space and let (rspaces) denote the category of locally ringed spaces. We define the *functor of points of the locally ringed space M* as the representable functor:

$$h_M\colon (\text{rspaces})^{\text{op}} \to (\text{sets}), \quad h_M(T) = \text{Hom}(T, M).$$

As before, h_M is defined on morphisms as follows: $h_M(\phi)(g) = g \circ \phi$.

If the locally ringed space M is a differentiable manifold, we have the following important characterization of morphisms.

Proposition 2.4.6. *Let M and N be differentiable manifolds. Then*

$$\text{Hom}(M, N) \cong \text{Hom}(C^\infty(N), C^\infty(M)).$$

We are going to see a proof of this result in the more general setting of supergeometry in Chapter 4, Section 4.5.

We are now going to state Yoneda's lemma, a basic categorical result. As an immediate consequence, we have that the functor of points of a scheme (resp. differentiable manifold) does determine the scheme (resp. differentiable manifold) itself.

Theorem 2.4.7 (Yoneda's lemma). *Let $\mathcal{C}$ be a category and let X, Y be objects in $\mathcal{C}$ and let $h_X : \mathcal{C}^{\mathrm{op}} \to$ (sets) be the representable functor defined on the objects as $h_X(T) = \mathrm{Hom}(T, X)$ and, as usual, on the arrows as $h_X(\phi)(f) = f \cdot \phi$, for $\phi : T \to S$, $f \in \mathrm{Hom}(T, X)$.*

(1) *If $F : \mathcal{C}^{\mathrm{op}} \to$ (sets), then we have a one-to-one correspondence between the sets:*

$$\{h_X \to F\} \iff F(X).$$

(2) *The functor*

$$h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathrm{sets})), \quad X \mapsto h_X,$$

is an equivalence of $\mathcal{C}$ with a full subcategory of functors. In particular, $h_X \cong h_Y$ if and only if $X \cong Y$ and the natural transformations $h_X \to h_Y$ are in one-to-one correspondence with the morphisms $X \to Y$.

Proof. We briefly sketch it, leaving the details to the reader. Let $\alpha : h_X \to F$. We can associate to α, $\alpha_X(\mathrm{id}_X) \in F(X)$. Vice versa, if $p \in F(X)$, we associate to p, $\alpha : h_X \to F$ such that

$$\alpha_Y : \mathrm{Hom}(Y, X) \to F(Y), \quad f \mapsto F(f)p. \qquad \square$$

Corollary 2.4.8. *Two schemes (resp. manifolds) are isomorphic if and only if their functors of points are isomorphic.*

The advantages of using the functorial language are many. Morphisms of schemes are just maps between the sets of their A-points, respecting functorial properties. This often simplifies matters, allowing us to leave the sheaves machinery in the background. The problem with such an approach, however, is that not all the functors from (schemes) to (sets) are the functors of points of a scheme, i.e., they are representable. The next theorem establishes an important criterion. We shall state the theorem for functors from (rings) to (sets) leaving to the reader, as a simple exercise, to write a similar statement for functors from (schemes) to (sets).

Theorem 2.4.9. *A functor $\mathcal{F}$: (rings) $\to$ (sets) is of the form $\mathcal{F}(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, X)$ for a scheme X if and only if:*

(1) *$\mathcal{F}$ is **local** or is a **sheaf in the Zariski topology**. This means that for each ring R and for every collection $\alpha_i \in \mathcal{F}(R_{f_i})$, with $(f_i, i \in I) = R$, so that α_i and α_j map to the same element in $\mathcal{F}(R_{f_i f_j})$ for all i and j, there exists a unique element $\alpha \in \mathcal{F}(R)$ mapping to each α_i.*

(2) $\mathcal{F}$ ***admits a cover by open affine subfunctors***. *This means that there exists a family* $\mathcal{U}_i$ *of subfunctors of* $\mathcal{F}$*, i.e.,* $\mathcal{U}_i(R) \subset \mathcal{F}(R)$ *for all* $R \in$ (rings), $\mathcal{U}_i = h_{\underline{\mathrm{Spec}}\, U_i}$*, with the property that for all natural transformations* $f: h_{\underline{\mathrm{Spec}}\, A} \to F$*, the functors* $f^{-1}(\mathcal{U}_i)$*, defined as* $f^{-1}(\mathcal{U}_i)(R) = f^{-1}(\mathcal{U}_i(R))$*, are all representable, i.e.,* $f^{-1}(\mathcal{U}_i) = h_{V_i}$*, and the* V_i *form an open covering of* $\underline{\mathrm{Spec}}\, A$.

Proof. See [29], p. 259 or [23], Ch. I. □

In Chapter 10 we are going to see a complete proof of this statement in the more general setting of superschemes.

This theorem states the conditions we expect for F to be the functor of points of a scheme. Namely, locally, F must look like the functor of points of a scheme (property (2)), moreover F must be a sheaf, that is F must have a gluing property that allows us to patch together the open affine cover we are given in the hypothesis (property (1) and (2) together).

A similar criterion holds for the functor of points in the differential category and more general for a locally ringed space and we are going to discuss it in Chapter 9.

We conclude this section by examining the two important examples of projective space and Grassmannian variety using the functorial point of view.

Examples 2.4.10. (1) *Projective space*. Let us revisit Example 2.3.4 of projective space as a scheme using the functor of points point of view. Define the functor h: (rings) $\to$ (sets), where $h(A)$ is the set of projective submodules of A^{n+1} of rank n. Equivalently, by duality, we have that $h(A)$ consists of the morphisms $\alpha: A^{n+1} \to L$, where L has rank 1, modulo the equivalence relation $\alpha \sim \alpha'$ if and only if $\ker(\alpha) = \ker(\alpha')$.

To complete the definition we need to specify h on morphisms $\psi: A \to B$.

Given a morphism $\psi: A \to B$, we can give to B an A-module structure by setting

$$a \cdot b = \psi(a)b, \quad a \in A,\ b \in B.$$

Also, given an A-module L, we can construct the B-module $L \otimes_A B$. So given ψ and the element of $h(A)$, $f: A^m \to L$, we have an element of $h(B)$,

$$h(\psi)(f): B^m \cong A^m \otimes_A B \to L \otimes_A B.$$

We want to show that h is the functor of points of the projective space, in other words $h = h^a_{\mathbb{P}^n}$. We briefly sketch how to check property (2) in Theorem 2.4.9; as for property (1) and in general for more details, we refer the reader to [23], Ch. 1, §1.

Property (2) says that we need to cover h by open affine subfunctors v_i. The functors v_i are defined as follows. For A local, $v_i(A)$ is the set of α such that $\alpha(0, \dots, a_i, \dots, 0)$ is invertible. As an exercise one can show that v_i corresponds to an open affine subfunctor of h, and it corresponds to the functor of points of an affine space of dimension n. If A is local we have the following nice characterization of the A-points of the projective space (see [29], Ch. III, §2).

Proposition 2.4.11. *The A-points of $\mathbb{P}^n$, for A local, are in one-to-one correspondence with the set of $n+1$-uples $[a_0, \dots, a_n] \in A^{n+1}$ such that at least one of the a_i is a unit, modulo the equivalence relation $[a_0, \dots, a_n] \cong [\lambda a_0, \dots, \lambda a_n]$ for any unit λ in A.*

(2) *Grassmannian scheme*. Classically the Grassmannian variety is a natural generalization of the projective space: its points are the set of r vector spaces inside an m-dimensional vector space ($r < m$). It is a projective variety and now we want to describe the functor of points of its associated scheme.

Define the functor $\mathrm{Gr}\colon$ (rings) $\to$ (sets) as follows. $\mathrm{Gr}(A)$ is the set of projective quotients of rank r of A^m, that is,

$$\mathrm{Gr}(A) = \{\alpha\colon A^m \to L, \text{ with } L \text{ a projective A-module of rank } r\}/\cong.$$

α is a surjective morphisms, L' is the kernel of α and $A^m = L \oplus L'$, $\cong$ means up to isomorphism of L. Equivalently this is the set

$$\begin{aligned}\mathrm{Gr}(A) &= \{A^m/L', \text{ with } L' \text{ a projective submodule of } A^m \text{ of rank } m-r\}\\ &= \{\text{projective submodules } L \text{ of } A^m \text{ of rank } r\}.\end{aligned}$$

As before we have the following characterization of $\mathrm{Gr}(A)$ in case A is a local ring (projective A-modules are free). $\mathrm{Gr}(A)$ is the set of all free A-modules of rank r in A^m modulo isomorphism. If we have a ground field k and $A = k$ we can see immediately that $\mathrm{Gr}(k)$ are the points of the classical Grassmannian variety of r subspaces in k^m.

To complete the definition we need to specify Gr on morphisms $\psi\colon A \to B$ the same way we did before.

Given a morphism $\psi\colon A \to B$, we can give to B the structure of an A-module by setting

$$a \cdot b = \psi(a)b, \quad a \in A,\ b \in B.$$

Also, given an A-module L, we can construct the B-module $L \otimes_A B$. So given ψ and the element of $\mathrm{Gr}(A)$, $f\colon A^m \to L$, we have an element of $\mathrm{Gr}(B)$,

$$\mathrm{Gr}(\psi)(f)\colon B^m = A^m \otimes_A B \to L \otimes_A B.$$

The fact that Gr is the functor of points of a superscheme is again an application of Theorem 2.4.9. For the property (1), namely the fact that Gr is local, we send the reader to [23], Ch. I, §1. For the property (2), we have to give explicitly a cover by open affine subfunctors. Consider the multi-index $I = (i_1, \dots, i_r)$, $1 \le i_1 < \dots < i_r \le m$, and the map $\phi_I\colon A^r \to A^m$ defined by setting $\phi_I(x_1, \dots, x_r) = (y_1, \dots, y_m)$, with $y_{i_k} = x_k$ for $k = 1, \dots, m$ and $y_j = 0$ otherwise.

Let us define subfunctors v_I of Gr for A local as follows:

$$v_I(A) = \{\alpha\colon A^m \to L \mid \alpha \circ \phi_I \text{ is invertible}\}.$$

We leave it to the reader, as an exercise, to verify that the v_I correspond to open affine subfunctors of Gr isomorphic to the functor of points of affine spaces. In any case, we are going to study in detail this same example in the supergeometric setting in Chapter 10. For more details on this and the previous example in the ordinary setting, see [23], Ch. I, §1, and [29], Ch. VI.

2.5 Coherent sheaves

In this section we briefly give the definition and some basic properties of coherent sheaves. This notion is introduced in ordinary algebraic geometry in order to characterize sheaves which have good properties and in general are well behaved. As we shall see in the next chapter, it is one of the building blocks for our definition of superscheme.

Let A be a commutative ring and M an A-module. We want to define a sheaf $\widetilde{M}$ on $\underline{\mathrm{Spec}}\, A$, which has an $\mathcal{O}_A$-module structure, i.e., for all open sets U in $\underline{\mathrm{Spec}}\, A$, we want $\widetilde{M}(U)$ to have an $\mathcal{O}_A(U)$-module structure compatible with the restriction morphisms. We are going to define the sheaf on the basic open sets U_f introduced in the previous section; by Proposition 2.2.11, this will suffice.

Let us consider the assignment

$$U_f \mapsto M_f,$$

where $M_f = M[f^{-1}]$ is the A_f-module obtained by M by inverting just the element $f \in A$. This assignment defines a $\mathcal{B}$-sheaf that, by Proposition 2.2.11, extends uniquely to a sheaf on $\underline{\mathrm{Spec}}\, A$ that we denote by $\widetilde{M}$. The next proposition summarizes the properties of $\widetilde{M}$.

Proposition 2.5.1. *Let M be a module for a commutative algebra A. The sheaf $\widetilde{M}$ defined above has the following properties:*

(1) *$\widetilde{M}$ is an $\mathcal{O}_A$-module;*

(2) *$(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\, A$, i.e., the stalk at any prime $\mathfrak{p}$ of the sheaf $\widetilde{M}$ coincides with the localization of M at $\mathfrak{p}$;*

(3) *$(\widetilde{M})(\mathrm{Spec}\, A) = M$, i.e., the global sections of the sheaf coincide with the A-module M.*

Proof. See Proposition 5.1, Ch. II in [43]. □

Definition 2.5.2. Let $X = (|X|, \mathcal{O}_X)$ be a scheme, $\mathcal{F}$ a sheaf on $|X|$ of $\mathcal{O}_X$-modules. In other words, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for all U open in $|X|$ and the restriction morphisms behave nicely with respect to the $\mathcal{O}_X$-module structure. We say $\mathcal{F}$ is *quasi-coherent*, if there exists an open affine cover $\{U_i = \underline{\mathrm{Spec}}\, A_i\}_{i \in I}$ of X such that $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$ for a suitable A_i-module M_i. $\mathcal{F}$ is *coherent* if the affine cover can be chosen so that the M_i's are finitely generated A_i-modules.

We are going to give an example of a quasi-coherent sheaf, which is most important for our algebraic supergeometry applications.

Example 2.5.3. Let R be a commutative super ring. R_0 is a commutative ring in the ordinary sense. Since R_1 is an R_0-module, the whole of R is also an R_0-module, hence we can construct the quasi-coherent sheaf $\widetilde{R}$ on the topological space $\operatorname{Spec} R_0$. One can easily check that $(\operatorname{Spec} R_0, \widetilde{R})$ is a locally ringed space. We shall see in more detail in Chapter 10 that this is the local model for our definition of superscheme.

The next proposition establishes an important equivalence of categories.

Proposition 2.5.4. *Let A be a commutative ring. The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent sheaves of $\mathcal{O}_A$-modules. The inverse of this functor is the functor $\mathcal{F} \mapsto \mathcal{F}(\operatorname{Spec} A)$. If A is noetherian, the same functor gives an equivalence of categories between the category of finitely generated A-modules and the coherent sheaves which are $\mathcal{O}_A$-modules.*

Proof. See Corollary 5.5, Ch. II in [43]. □

2.6 References

For the theory of sheaves and schemes see [37], [43], [29], [38] and the fundamental paper by Serre that originated the theory [70]. For a complete treatment of the functor of points of a scheme see [23]; however, a good summary of most of the properties needed is found in [29]. As for all the algebraic statements (e.g. the definition of localization, its properties and so on), we refer to the classical textbooks [51], [2].

3

Supergeometry

In this introductory chapter, we begin our discussion on the foundations of supergeometry. We give an overview on some fundamental supergeometric objects, namely supermanifolds and superschemes, that we shall discuss in detail in the forthcoming chapters. Here, we also want to stress how the functorial treatment allows the simultaneous treatment of both the differential and the algebraic categories at once.

The most basic supergeometric object, the *superspace*, is first introduced, from which the concepts of *supermanifolds* and *superschemes* are then built. We leave for later chapters the full development of all of these notions. We also give a brief description of the functor of points approach in supergeometry, which will be massively employed, both in the differential and algebraic category, in all of the following chapters. We illustrate the definitions and the basic concepts with some key examples, including supermatrices and the general linear supergroup, which turn out to be fundamental in the later development.

3.1 Superspaces

A unified way to look at the categories of ordinary differentiable manifolds or algebraic schemes, is to think of an object as a pair, consisting of a topological space together with a sheaf of functions defined on it. Such a pair is often referred to as a *ringed space*. For ordinary manifolds, for example, the sheaf of functions is the sheaf of the C^∞ functions while for ordinary algebraic varieties it is the sheaf of regular functions, as we have seen in the previous chapter. We are going to generalize this point of view, discussed in detail for the ordinary setting in the previous chapter, introducing supermanifolds and superschemes in the framework of ringed spaces.

Definition 3.1.1. As in ordinary algebraic geometry, a *super ringed space* S is a topological space $|S|$ endowed with a sheaf of commutative super rings, called the *structure sheaf* of S, which we denote by $\mathcal{O}_S$. Let S denote the super ringed space $(|S|, \mathcal{O}_S)$.

Notice that $S_0 := (|S|, \mathcal{O}_{S,0})$ is an ordinary ringed space as in Definition 2.2.13, where $\mathcal{O}_{S,0}(U) := \mathcal{O}_S(U)_0$ is a sheaf of ordinary rings on $|S|$. Notice also that $\mathcal{O}_{S,1}(U) := \mathcal{O}_S(U)_1$ defines a sheaf of $\mathcal{O}_{S,0}$-modules on $|S|$, i.e., for all open sets U in $|S|$, we have that $\mathcal{O}_{S,1}(U)$ is an $\mathcal{O}_{S,0}(U)$-module and this structure is compatible with the restriction morphisms.

Definition 3.1.2. A *superspace* is a super ringed space S with the property that the stalk $\mathcal{O}_{S,x}$ is a local super ring for all $x \in |S|$.

As in the ordinary setting a commutative super ring is *local* if it has a unique maximal ideal. Notice that any prime ideal in a commutative super ring must contain the whole odd part since it contains all nilpotents.

As we have seen in the previous chapter, ordinary differentiable manifolds and algebraic schemes are examples of superspaces, where we think of their sheaves of functions as sheaves of commutative super rings with trivial odd part.

Let us now see an example of a superspace with non-trivial odd part.

Example 3.1.3. Let M be a differentiable manifold, $|M|$ its underlying topological space, C_M^∞ the sheaf of ordinary C^∞ functions on M. We define the sheaf of supercommutative $\mathbb{R}$-algebras as (for $V \subset |M|$ open)

$$V \mapsto \mathcal{O}_M(V) := C_M^\infty(V)[\theta^1, \dots, \theta^q],$$

where $C_M^\infty(V)[\theta^1, \dots, \theta^q] = C_M^\infty(V) \otimes \wedge(\theta^1, \dots, \theta^q)$ and the θ^j are odd (anticommuting) indeterminates. As one can readily check, $(|M|, \mathcal{O}_M)$ is a super ringed space; moreover $(|M|, \mathcal{O}_M)$ is also a superspace. In fact $\mathcal{O}_{M,x}$ is a local ring, with maximal ideal $m_{M,x}$ generated by the maximal ideal of the local ring $C_{M,x}^\infty$ and the odd elements $\theta^1, \dots, \theta^q$. One can check immediately that all the elements in $\mathcal{O}_{M,x} \setminus m_{M,x}$ are invertible.

In the special case $M = \mathbb{R}^p$, we define the superspace

$$\mathbb{R}^{p|q} = (\mathbb{R}^p, C_{\mathbb{R}^p}^\infty[\theta^1, \dots, \theta^q]).$$

From now on, with an abuse of notation, $\mathbb{R}^{p|q}$ denotes both the super vector space $\mathbb{R}^p \oplus \mathbb{R}^q$ and the superspace $(\mathbb{R}^p, C_{\mathbb{R}^p}^\infty[\theta^1, \dots, \theta^q])$, the context making clear which one we mean. $\mathbb{R}^{p|q}$ plays a key role in the definition of supermanifold since it is the local model. If $t^1, \dots, t^p$ are global coordinates for $\mathbb{R}^p$ we shall speak of $t^1, \dots, t^p, \theta^1, \dots, \theta^q$ as a set of *global coordinates* for the superspace $\mathbb{R}^{p|q}$.

We are going to study the example of $\mathbb{R}^{p|q}$ in detail in the next chapter.

Definition 3.1.4. Let $S = (|S|, \mathcal{O}_S)$ be a superspace. Given an open subset $|U| \subset |S|$, the pair $(|U|, \mathcal{O}_S|_{|U|})$ is always a superspace, called the *open subspace* associated to $|U|$.

The next example is very important for our applications.

Example 3.1.5 (Supermatrices $\mathrm{M}_{p|q}$ and the general linear supergroup $\mathrm{GL}_{p|q}$). Let $\mathrm{M}_{p|q} = \mathbb{R}^{p^2+q^2|2pq}$. This is the superspace corresponding to the super vector space of $(p|q \times p|q)$-matrices, the underlying topological space being $\mathrm{M}_p \times \mathrm{M}_q$, the direct product of $p \times p$ and $(q \times q)$-matrices (see also Remark 1.4.1). As a super vector space we have

$$\mathrm{M}_{p|q} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}, \quad (\mathrm{M}_{p|q})_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad (\mathrm{M}_{p|q})_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\},$$

where A, B, C, D are respectively $p \times p$, $p \times q$, $q \times p$, $(q \times q)$-matrices with entries in $\mathbb{R}$.

Hence as a superspace $\mathrm{M}_{p|q}$ has $p^2 + q^2$ even global coordinates t^{ij}, $1 \leq i, j \leq p$ or $p+1 \leq i, j \leq p+q$, and $2pq$ odd ones θ^{kl}, $1 \leq k \leq p$, $p+1 \leq l \leq p+q$ or $p+1 \leq k \leq p+q$, $1 \leq l \leq p$. The structure sheaf of $\mathrm{M}_{p|q}$ is the assignment

$$V \mapsto \mathcal{O}_{\mathrm{M}_{p|q}}(V) = C^\infty_{\mathrm{M}_p \times \mathrm{M}_q}(V)[\theta^{kl}] \quad \text{for all } V \text{ open in } \mathrm{M}_p \times \mathrm{M}_q.$$

The superspace $\mathrm{M}_{p|q} = (\mathrm{M}_p \times \mathrm{M}_q, \mathcal{O}_{\mathrm{M}_{p|q}})$ is called the superspace of *supermatrices*. Now let us consider in the topological space $\mathrm{M}_p \times \mathrm{M}_q = \mathbb{R}^{p^2+q^2}$, the open set U consisting of the points for which $\det(t_{ij})_{1 \leq i,j \leq p} \neq 0$ and $\det(t_{ij})_{p+1 \leq i,j \leq p+q} \neq 0$. We define the superspace $\mathrm{GL}_{p|q} := (U, \mathcal{O}_{\mathrm{M}_{p|q}}|_U)$, the open subspace of $\mathrm{M}_{p|q}$ associated to the open set U. As we shall see, this superspace has a Lie supergroup structure and it is called the *general linear supergroup*.

Next we define morphisms of superspaces, so that we can talk about the category of superspaces.

Definition 3.1.6. Let S and T be superspaces. Then a morphism $\varphi\colon S \to T$ is a continuous map $|\varphi|\colon |S| \to |T|$ together with a sheaf morphism $\varphi^*\colon \mathcal{O}_T \to \varphi_* \mathcal{O}_S$ so that $\varphi^*_x(\mathfrak{m}_{T,|\varphi|(x)}) \subset \mathfrak{m}_{S,x}$, where $\mathfrak{m}_{S,x}$ is the maximal ideal in $\mathcal{O}_{S,x}$, while $\mathfrak{m}_{T,|\varphi|(x)}$ is the maximal ideal in $\mathcal{O}_{T,|\varphi|(x)}$ and φ^*_x is the stalk map.

Remark 3.1.7. Recall from the previous chapter that the sheaf morphism $\varphi^*\colon \mathcal{O}_T \to \varphi_* \mathcal{O}_S$ corresponds to the system of maps $\varphi^*_U\colon \mathcal{O}_T(U) \to \mathcal{O}_S(|\varphi|^{-1}(U))$ for all open sets $U \subset |T|$, compatible with the restriction morphisms. To ease notation, we also refer to the maps φ^*_U as φ^*.

Essentially the condition $\varphi^*_x(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$ means that the sheaf homomorphism is local. Note also that φ^* is a morphism of supersheaves, so, as usual, it preserves the parity. The main point to make here is that when we are giving a morphism of superspaces the sheaf morphism must be specified along with the continuous topological map, since sections are not necessarily genuine functions on the topological space as in ordinary differential geometry. An arbitrary section cannot be viewed as a function because commutative super rings have many nilpotent elements, and nilpotent sections are identically zero as functions on the underlying topological space. Therefore we employ the methods of algebraic geometry to study such objects. We will address this in more detail later. Now we introduce two types of superspaces that we examine in detail in the forthcoming chapters: supermanifolds and superschemes.

3.2 Supermanifolds

A supermanifold is a specific type of superspace, which we describe via a local model. Because we always keep an eye on the physics literature from which supersymmetry springs, the supermanifolds of interest to us are the C^∞-supermanifolds over $\mathbb{R}$. Nevertheless, all the definitions we give here hold also in the context of analytic real or complex supermanifolds, as we shall see more explicitly at the end of the next chapter.

Let C^∞_U be the sheaf of C^∞-functions on the domain $U \subset \mathbb{R}^p$. We define the *superdomain* $U^{p|q}$ to be the superspace $(U, C^\infty_U[\theta^1, \dots, \theta^q])$ where $C^\infty_U[\theta^1, \dots, \theta^q] = C^\infty_{\mathbb{R}^p}|_U \otimes \bigwedge(\theta^1, \dots, \theta^q)$. Most immediately, the superspaces $\mathbb{R}^{p|q}$ are superdomains with sheaf $C^\infty_{\mathbb{R}^p}[\theta^1, \dots, \theta^q]$.

Definition 3.2.1. A *supermanifold* $M = (|M|, \mathcal{O}_M)$ of dimension $p|q$ is a superspace that is locally isomorphic to $\mathbb{R}^{p|q}$. In other words, given any point $x \in |M|$, there exists a neighborhood $V \subset |M|$ of x with q odd indeterminates θ^j so that

$$V \cong V_0 \text{ open in } \mathbb{R}^p, \quad \mathcal{O}_M|_V \cong \underbrace{C^\infty(t^1, \dots, t^p)}_{C^\infty_{\mathbb{R}^p}|_{V_0}}[\theta^1, \dots, \theta^q]. \tag{3.1}$$

We call $t^1, \dots, t^p, \theta^1, \dots, \theta^q$ the *local coordinates* of M in V and $p|q$ the *superdimension* of the supermanifold M.

Morphisms of supermanifolds are morphisms of the underlying superspaces. For M, N supermanifolds, a morphism $\varphi\colon M \to N$ is a continuous map $|\varphi|\colon |M| \to |N|$ together with a local morphism of sheaves of superalgebras $\varphi^*\colon \mathcal{O}_N \to \varphi_* \mathcal{O}_M$, where *local*, as before (see Chapter 2), means that $\varphi_x^{-1}(m_{M,x}) = m_{N,|\varphi|(x)}$, where $\varphi_x\colon \mathcal{O}_{N,|\varphi|(x)} \to \mathcal{O}_{M,x}$ is the stalk morphism, for a point $x \in |M|$, and $m_{M,x}, m_{N,|\varphi|(x)}$ are the maximal ideals in the stalks. Note that in the purely even case of ordinary C^∞-manifolds, the above notion of a morphism agrees with the ordinary one.

We may therefore talk about the category of supermanifolds. The difficulty in dealing with C^∞-supermanifolds arises when one tries to think of "points" or "functions" in the traditional sense. The ordinary points only account for the topological space and the underlying sheaf of ordinary C^∞-functions, and one may truly only talk about the "value" of a section $f \in \mathcal{O}_M(U)$ for $U \subset |M|$ an open subset; the value of f at $x \in U$ is the unique real number c so that $f - c$ is not invertible in any neighborhood of x. For concreteness, let us consider the example of $M = \mathbb{R}^{1|1}$, with global coordinates t and θ. Let us take the global section $f = t\theta \in \mathcal{O}_M(\mathbb{R})$. For any non-zero real number c, we have that $t\theta - c$ is always invertible since $t\theta$ is nilpotent, its inverse being $-c^{-2}t\theta - c^{-1}$. Hence the value of $t\theta$ at all points $x \in \mathbb{R} = |\mathbb{R}^{1|1}|$ is zero. What this says is that we cannot reconstruct a section by knowing only its values at topological points. Such sections are then not truly functions in the ordinary sense, however, now that we have clarified this point, we may adhere to the established notation and call such sections *"functions on U"*. We shall return to the delicate notion of value of a section at a point in Chapter 4, Section 4.1.

Remark 3.2.2. Let M be a supermanifold, U an open subset in $|M|$, and f a function on U. If $\mathcal{O}_M(U) = C^\infty(t^1, \dots, t^p)[\theta^1, \dots, \theta^q]$ as in (3.1), there exist even functions $f_I \in C^\infty(t)$ $(t = t^1, \dots, t^p)$) so that

$$f(t,\theta) = f_0(t) + \sum_i f_i(t)\theta^i + \sum_{i<j} f_{ij}(t)\theta^i\theta^j + \cdots = f_0(t) + \sum_{|I|=1}^{q} f_I(t)\theta^I, \quad (3.2)$$

where $I = \{i_1 < i_2 < \cdots < i_r\}_{r=1}^q$.

So in some sense, we can expand $f(t,\theta)$ in power series, with respect to the odd coordinates θ^j's. We are going to discuss this important point further in the next chapter.

Let us establish the following notation. Let M be a supermanifold, then we write the nilpotent sections (i.e., the sections in the nilpotent part of $\mathcal{O}_M$) as

$$J_M = \mathcal{O}_{M,1} + \mathcal{O}_{M,1}^2 = \langle \mathcal{O}_{M,1} \rangle_{\mathcal{O}_M}.$$

This is an ideal sheaf in $\mathcal{O}_M$ and defines a natural subspace of M we shall call M_{red} or $\widetilde{M}$ (not to be confused with the notation for coherent sheaves in Proposition 2.5.1), the *reduced manifold associated with* M where

$$\widetilde{M} = (|M|, \mathcal{O}_M / J_M).$$

Note that $\widetilde{M}$ is a completely even superspace, and hence lies in the ordinary category of ordinary C^∞-manifolds, i.e., it is locally isomorphic to $\mathbb{R}^p$. The quotient map from $\mathcal{O}_M \to \mathcal{O}_M / J_M$ defines the inclusion morphism $\widetilde{M} \hookrightarrow M$. The subspace $\widetilde{M}$ should not be confused with the purely even superspace $(|M|, \mathcal{O}_{M,0})$, which is *not* a C^∞-manifold, since the structure sheaf still contains nilpotents.

We now examine closed submanifolds in the super category.

Definition 3.2.3. Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold. We say that $N = (|N|, \mathcal{O}_M/\mathcal{I})$ is a *submanifold* of M if:

(1) N is a supermanifold,

(2) $|N|$ is a closed subset of $|M|$,

(3) $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_M$ with the following property: for all $x \in |N|$, there exists a neighbourhood U_x of x in $|M|$ and an appropriate choice of coordinates in M and N such that $t^1, \dots, t^m, \theta^1, \dots, \theta^n$ are coordinates for x in the open subspace $(U_x, \mathcal{O}_M|_{U_x})$ of M while $t^1, \dots, t^p, \theta^1, \dots, \theta^q$ are coordinates for x in the open subspace $(U_x \cap |N|, (\mathcal{O}_M/\mathcal{I})_{U_x \cap |N|})$ of N (where $p \le m$, $q \le n$).

One sees immediately that $\widetilde{M} = (|M|, \mathcal{O}_M/J_M)$ is a submanifold of M while there is no natural way to realize $(|M|, \mathcal{O}_{M,0})$ as a submanifold of M.

We shall return to the concept of submanifold in Chapter 5, Section 5.3.

3.3 Superschemes

A *superscheme* is an object in the category of superspaces that generalizes the notion of an ordinary scheme, which we have introduced and discussed in the previous chapter.

Definition 3.3.1. A superspace $S = (|S|, \mathcal{O}_S)$ is a *superscheme* if $(|S|, \mathcal{O}_{S,0})$ is an ordinary scheme and $\mathcal{O}_{S,1}$ is a quasi-coherent sheaf of $\mathcal{O}_{S,0}$-modules.

As we shall see in Chapter 10, superschemes are also characterized by local models, called in this case *affine superschemes*. They are superspaces locally isomorphic to the *spectrum of superrings*, in analogy with the ordinary setting.

Because any non-trivial supercommutative ring has non-zero nilpotents, we need to redefine what we mean by a reduced superscheme.

Definition 3.3.2. We say that a superscheme S is *super reduced* if $\mathcal{O}_S/J_S$ is reduced. In other words, in a super reduced superscheme, we want that the odd sections generate all the nilpotents.

Example 3.3.3 (The affine superspace). Let $\mathbb{A}^m$ be the ordinary affine space of dimension m over a field k. $\mathbb{A}^m$ consists of the topological space k^m, that is the vector space k^m with the Zariski topology, and the sheaf $\mathcal{O}_{\mathbb{A}^m}$ of regular functions on k^m. On k^m we define the sheaf $\mathcal{O}_{\mathbb{A}^{m|n}}$ of superalgebras in the following way. Given $U \subset k^m$ open,

$$\mathcal{O}_{\mathbb{A}^{m|n}}(U) = \mathcal{O}_{k^m}(U) \otimes \bigwedge(\xi_1, \dots, \xi_n)$$

where $\xi_1, \dots, \xi_n$ are odd (anticommuting) indeterminates. One may readily check that $(k^m, \mathcal{O}_{\mathbb{A}^{m|n}})$ is a reduced superscheme, which we here denote by $\mathbb{A}^{m|n}$.

Let us now revisit the Example 3.1.5 taking the algebraic point of view.

Example 3.3.4 (Algebraic supermatrices $\mathrm{M}^{\mathrm{alg}}_{p|q}$ and the algebraic general linear supergroup $\mathrm{GL}^{\mathrm{alg}}_{p|q}$). Let $\mathrm{M}^{\mathrm{alg}}_{p|q} = \mathbb{A}^{p^2+q^2|2pq}$ be the affine superspace corresponding to the super vector space of $(p|q \times p|q)$-matrices with entries in the field k. The underlying topological space of $\mathrm{M}^{\mathrm{alg}}_{p|q}$ is the product $\mathrm{M}^{\mathrm{alg}}_p \times \mathrm{M}^{\mathrm{alg}}_q$, where $\mathrm{M}^{\mathrm{alg}}_p$ denotes the set of $(p \times p)$-matrices with entries in k, with the Zariski topology. The super ring of global sections of $\mathrm{M}^{\mathrm{alg}}_{p|q}$ is $k[t^{ij}, \theta^{kl}]$, with $1 \leq i, j \leq p$ or $p+1 \leq i, j \leq p+q$ and $1 \leq k \leq p$, $p+1 \leq l \leq p+q$ or $p+1 \leq k \leq p+q$, $1 \leq l \leq p$. The conditions $\det(t_{ij})_{1 \leq i,j \leq p} \neq 0$ and $\det(t_{ij})_{p+1 \leq i,j \leq p+q} \neq 0$ define a Zariski open set U in $\mathrm{M}^{\mathrm{alg}}_p \times \mathrm{M}^{\mathrm{alg}}_q$, hence we have a superspace $\mathrm{GL}^{\mathrm{alg}}_{p|q} = (U, \mathcal{O}_{\mathrm{M}_{p|q}}|_U)$. One can check immediately that this is a superscheme. From now on we shall drop the suffix alg to improve readability, the context making clear if we are considering the general linear supergroup $\mathrm{GL}_{p|q}$ in the algebraic or in the differential context.

Morphisms of superschemes are just morphisms of superspaces, so we may talk about the subcategory of superschemes. The category of superschemes is larger than the category of schemes; any scheme is a superscheme if we take a trivial odd component in the structure sheaf. We will complete our exposition of the category of superschemes in Chapters 10–11.

3.4 The functor of points

The presence of odd coordinates steals some of the geometric intuition away from the language of supergeometry. For instance, we cannot see an "odd point" – they are invisible both topologically and as classical functions on the underlying topological space. We see the odd points only as sections of the structure sheaf. To bring some of the intuition back, we turn to the functor of points approach from algebraic geometry.

Definition 3.4.1. Let S and T be superspaces. A T*-point* of S is a morphism $T \to S$. We denote the set of all T-points by $S(T)$. Equivalently,

$$S(T) = \mathrm{Hom}(T, S).$$

We define the *functor of points* of the superspace S to be the functor

$$S\colon (\text{sspaces})^{\text{op}} \to (\text{sets}), \quad T \mapsto S(T), \quad S(\phi)(f) = f \circ \phi,$$

where (sspaces) denotes the category of superspaces and the superscript op as usual refers to the opposite category (see Appendix B for more details).

By a common abuse of notation the superspace S and the functor of points of S are denoted by the same letter. Whenever it is necessary to make a distinction, we shall write h_S for the functor of points of the superspace S.

We have defined the functor of points of a superspace; clearly we can also define the functor of points of a supermanifold or a superscheme, just by changing the category we start from.

Definition 3.4.2. Let (smflds) and (sschemes) denote respectively the categories of supermanifolds and superschemes introduced above. We define the *functor of points of the supermanifold* M to be the functor

$$M\colon (\text{smflds})^{\text{op}} \to (\text{sets}), \quad T \mapsto M(T), \quad M(\phi)f = f \circ \phi.$$

Similarly we define the *functor of points of the superscheme* X to be the functor

$$X\colon (\text{sschemes})^{\text{op}} \to (\text{sets}), \quad T \mapsto X(T), \quad X(\phi)f = f \circ \phi.$$

The importance of the functor of points is a consequence of the following lemma, which is one of the many versions of Yoneda's lemma (see also Theorem 2.4.7).

Lemma 3.4.3 (Yoneda's lemma). *Let M and N be two superspaces (resp. supermanifolds or superschemes). There is a bijection from the set of morphisms $\varphi\colon M \to N$ to the set of maps $\varphi_T : M(T) \to N(T)$, functorial in T. In particular M and N are isomorphic if and only if their functors of points are isomorphic.*

Proof. Given a map $\varphi\colon M \to N$, for any morphism $t\colon T \to M$, $\varphi \circ t$ is a morphism $T \to N$. Conversely, we attach to the system (φ_T) the image of the identity map from $\varphi_M : M(M) \to N(M)$. For more details see Appendix B. □

Yoneda's lemma allows us to replace a superspace (resp. a supermanifold or a superscheme) S with its set of T-points, $S(T)$. We can now think of S as a representable functor from the category of superspaces (resp. supermanifolds or superschemes) to the category of sets. In fact, when constructing a superspace, it is often most convenient to construct first its functor of points and then prove that the functor is *representable* in the appropriate category. In fact in Chapter 9 we shall give a criterion for the representability of functors from the category of supermanifolds to the category of sets while in Chapter 10 we shall prove the same result for superschemes. As in the ordinary case (see Theorem 2.4.9), it turns out that representable functors must be *local*, i.e., they should admit a cover by open affine subfunctors, which glue together in some sense that we shall specify.

The following proposition, that we shall prove in the next chapter, is very useful when we want to explicitly describe the functor of points of a supermanifold or a superscheme. Its very formulation shows how the functorial treatment allows us to deal at once with both the differential and algebraic categories.

Remark 3.4.4. To ease the notation we write $\mathcal{O}_T(T)$ or simply $\mathcal{O}(T)$ for the global sections of a superspace T.

Proposition 3.4.5. *Let $M = (|M|, \mathcal{O}_M)$ and $T = (|T|, \mathcal{O}_T)$ be supermanifolds or affine superschemes. Then*

$$\mathrm{Hom}(T, M) = \mathrm{Hom}(\mathcal{O}(M), \mathcal{O}(T)).$$

Let us give some examples of T-points.

Examples 3.4.6. (i) Let T be just an ordinary topological point viewed as a supermanifold, i.e., $T = \mathbb{R}^{0|0} = (\mathbb{R}^0, \mathbb{R})$. By definition a T-point of a manifold M is a morphism $\phi\colon \mathbb{R}^{0|0} \to M$. ϕ consists of a continuous map $|\phi|\colon \mathbb{R}^0 \to |M|$, which corresponds to the choice of a point x in the topological space $|M|$, and a sheaf morphism $\phi^*\colon \mathcal{O}_M \to \phi_*(\mathbb{R})$, which assigns to a section its value in x. Then a T-point of M is an ordinary topological point of $|M|$.

(ii) Let M be the supermanifold $\mathbb{R}^{p|q}$ and let T be a supermanifold. By the previous proposition we have that a T-point of M corresponds to a morphism

$$\mathcal{O}(M) = C^\infty(t^1, \ldots, t^p)[\theta^1, \ldots, \theta^q] \to \mathcal{O}(T).$$

Then, in this case, a T-point of M is a choice of p even and q odd global sections on T. This is made more clear in Chapter 4 by Theorem 4.1.11. Thus $\mathbb{R}^{p|q}(T) = \mathcal{O}_{T,0}^{p}(T) \oplus \mathcal{O}_{T,1}^{q}(T)$.

(iii) Let X be the superscheme $\mathbb{A}^{m|n}$ as in Example 3.3.3 and let T be an affine superscheme. By definition, a T-point of X is a morphism of schemes $\phi\colon T \to \mathbb{A}^{m|n}$, which again, by the previous proposition, corresponds to a super ring morphism $\psi\colon \mathcal{O}(\mathbb{A}^{m|n}) \to \mathcal{O}(T)$, that is $\psi\colon k[x_1, \dots, x_m, \xi_1, \dots, \xi_n] \to \mathcal{O}(T)$, where $k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$ denotes the polynomial algebra in the even (commuting) indeterminates $x_1, \dots, x_m$ and in the odd (anticommuting) ones $\xi_1, \dots, \xi_n$. Hence, as before, ψ amounts to a choice of m even global sections in $\mathcal{O}(T)$ and n odd ones:

$$\begin{aligned}\mathbb{A}^{m|n}(T) &= \mathcal{O}(T)_0^m \oplus \mathcal{O}(T)_1^n \\ &= \{(a_1, \dots, a_m, \alpha_1, \dots, \alpha_n) \mid a_i \in \mathcal{O}(T)_0,\ \alpha_j \in \mathcal{O}(T)_1\}.\end{aligned}$$

We already see the power of T-points in these examples. The first example ($T = \mathbb{R}^{0|0}$) gives us complete topological information while the second ($M = \mathbb{R}^{p|q}$) allow us to talk about coordinates on supermanifolds. The third example shows how remarkably the differential and the algebraic categories resemble each other under the functorial treatment.

We plan to fully explore all these topics in the next chapters.

3.5 References

For our brief introduction and overview of supergeometry we send the reader to the works by Berezin [10], Kostant [49], Manin [56] and the notes of Bernstein [22]. In such papers, especially at the beginning, such exposition is done with more details. In particular in [56] there is a discussion of what a superscheme is, and, though implicitly, the problem of representability makes its appearance. In [49], in 2.15, there is a discussion of Proposition 3.4.5.

4

Differentiable supermanifolds

In this chapter we come to a more systematic and detailed study of supermanifolds. We shall be mainly interested in smooth supermanifolds, briefly discussing, only towards the end of the chapter, some aspects of real and complex analytic supermanifolds.

As we have seen in the previous introductory chapter, a smooth supermanifold is a superspace locally isomorphic to the superspace $\mathbb{R}^{m|n}$, which is an example of *superdomain*. We start by discussing various results and local properties of superdomains; the most important is the Chart Theorem, which allows us to characterize morphisms of superdomains.

We then proceed to a thorough description of the category of supermanifolds and their morphisms, introducing the concepts of tangent space and differential of a supermanifold morphism.

In analogy with ordinary differential geometry, we shall see that we can reconstruct a supermanifold starting from the superalgebra of the global sections of its structure sheaf. This is a fundamental result, and for this reason we devote a good part of this chapter to proving it in all details. As a consequence of this result we have a bijective (contravariant) correspondence between morphisms of supermanifolds and morphisms of the superalgebras of their global sections. This is the main tool we are going to employ to discuss supermanifold theory via the functor of points approach.

4.1 Superdomains and their morphisms

In this section we collect a few fundamental results on superdomains that we shall need in the sequel to develop the theory of smooth supermanifolds. The main result here is the *Chart Theorem*, which allows us to identify a morphism between superdomains of dimension $m|n$ and $p|q$ with p even and q odd functions in m even and n odd indeterminates. This is a very natural generalization of a similar result for ordinary differentiable manifolds.

Let us start by recalling a few definitions from Chapter 3.

Definition 4.1.1. We say that $S = (|S|, \mathcal{O}_S)$ is a *super ringed space* if $|S|$ is a topological space and $\mathcal{O}_S$ is a sheaf of super rings. If the stalk $\mathcal{O}_{S,x}$ is a local super ring for all points $x \in |S|$ we say that S is a *superspace*.

Definition 4.1.2. We define the smooth *superdomain* of dimension $(p|q)$ to be the super ringed space $U^{p|q} = (U, \mathcal{O}_{U^{p|q}})$, where U is an open subset of $\mathbb{R}^p$ and $\mathcal{O}_{U^{p|q}}(V) =$

$C^\infty(V) \otimes \bigwedge^q$ for each V open subset of U, with $\bigwedge^q = \bigwedge(\theta^1, \dots, \theta^q)$. $p|q$ is called the *superdimension* of the superdomain. We denote by $\mathcal{O}(U^{p|q})$ the global sections of the sheaf $\mathcal{O}_{U^{p|q}}$.

A *morphism* between two superdomains $V^{n|m}$ and $U^{p|q}$ is a morphism of super ringed spaces.

Remark 4.1.3. It is a consequence of Lemma 4.1.9 or of a similar ordinary result (see Example 3.1.3) that each superdomain is actually a superspace. Moreover we will show that morphisms between smooth superdomains are automatically superspace morphisms (see Lemma 4.2.3).

In the following we will make frequent use of the multi-index notation. An n-tuple $I = (i_1, \dots, i_n)$ of non-negative integers is called a multi-index. We define $|I| := \sum_{j=1}^n i_j$ and $I! := i_1! i_2! \dots i_n!$. Moreover we will use the shorthand

$$\frac{\partial^{|I|}}{\partial x^I} := \frac{\partial^{|I|}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}.$$

If $f \in \mathcal{O}_{U^{p|q}}(V)$, we can speak of the *value* $f(x)$ of f at a point $x \in V$.

Definition 4.1.4. Let $f = f_0 + \sum_{|I| \ge 1} f_I \theta^I$ be a section in $\mathcal{O}_{U^{p|q}}(V)$, f_0, $f_I \in C^\infty(V)$. If $x \in V$, the *value* $f(x)$ of f at x is $f_0(x)$.

Remark 4.1.5. When we speak of the *value* of a section we mean a real number $f(x) \in \mathbb{R}$. We may however use the same notation to mean just a section in $\mathcal{O}_{U^{p|q}}(V)$, to stress its dependence on x, which hence in this case is not assumed to be a point, but just a set of coordinates. This ambiguity is already present in the ordinary setting, and we shall make an effort to warn the reader whenever confusion may arise.

The next lemma gives an intrinsic characterization of $f(x)$.

Lemma 4.1.6. *Let $f = f_0 + \sum_{|I| \ge 1} f_I \theta^I$ be a section in $\mathcal{O}_{U^{p|q}}(V)$. If $x \in V$, then $f(x)$ is the unique number such that $f - f(x)$ is not invertible in any neighborhood of x contained in V.*

Proof. Classically, that is when $f = f_0$, the result is true, hence $f_0(x)$ is the unique real number for which $f_0 - f_0(x)$ is not invertible in any neighbourhood of x. We have $f - f_0(x) = f_0 - f_0(x) + n$, with n nilpotent. Since invertibility is not changed by adding a nilpotent element, we have that $f - f_0(x)$ is not invertible in any neighbourhood of x. Uniqueness also comes from the classical result. □

If $U^{p|q}$ is an ordinary domain, that is $q = 0$, the value of a section f, now a true function on the topological space U, is $f(x)$, that is, the function f evaluated at the point x as one usually intends this.

$\mathbb{R}^{p|q}$ denotes the superdomain whose reduced space is $\mathbb{R}^p$ and whose sheaf of sections is given by $\mathcal{O}_{\mathbb{R}^{p|q}}(U) := C^\infty(U) \otimes \wedge^q$ for each U open subset of $\mathbb{R}^p$.

Remark 4.1.7. The reader should notice that here $\mathbb{R}^{p|q}$ denotes a completely different object than the super vector space $\mathbb{R}^{p|q}$ introduced in Chapter 1. We will see in Example 4.6.3, in what sense these two objects can be identified.

If $\{t^i\}_{i=1}^p$ are coordinates in $C^\infty(U)$, and $\{\theta^j\}_{j=1}^q$ is a system of linearly independent algebraic generators of $\bigwedge^q$, then the set $\{t^i, \theta^j\}$ is called a *system of* (*super*) *coordinates* on $U^{p|q}$. The assignment of a superdomain $U^{p|q}$ together with a system of super coordinates is called a *superchart* or *chart* for short. We notice that on $U^{p|q}$ there is a *canonical chart*, consisting of the canonical coordinates inherited from $\mathbb{R}^{p|q}$.

We now want to discuss morphisms between superdomains in more detail. We start with an example, that will lead us to the formulation of the Chart Theorem.

Example 4.1.8. Consider the supermanifold $\mathbb{R}^{1|2}$ with a morphism $\phi\colon \mathbb{R}^{1|2} \to \mathbb{R}^{1|2}$.

On $\mathbb{R}^{1|2}$ we have global coordinates t, θ^1, θ^2 and so we may express any section f as in (3.2):

$$f = f(t, \theta^1, \theta^2) = f_0(t) + f_1(t)\theta^1 + f_2(t)\theta^2 + f_{12}(t)\theta^1\theta^2.$$

Then $f_0 \in C^\infty(\mathbb{R})$ sits as a function on the C^∞-manifold $\mathbb{R}$. By definition the morphism ϕ is described by a continuous map $|\phi|$ and a sheaf morphism ϕ^*.

Let us first prescribe the images of the global coordinates under ϕ^*:

$$\begin{aligned} t &\mapsto t^* := t + \theta^1\theta^2, \\ \theta^1 &\mapsto \theta^{1*} := \theta^1, \\ \theta^2 &\mapsto \theta^{2*} := \theta^2. \end{aligned} \tag{4.1}$$

We claim that knowing ϕ^* on only these global coordinates is enough to completely describe ϕ. Indeed, we first see that $t \mapsto t + \theta^1\theta^2$ tells us that $|\phi|$ is just the identity map. Next, let $f \in C^\infty(t)[\theta^1, \theta^2]$ be as above. Then $f \mapsto \phi^*(f) := f^*$ can be written formally as

$$\begin{aligned} f^* &= f(t^*, \theta^{1*}, \theta^{2*}) \\ &= f_0(t + \theta^1\theta^2) + f_1(t + \theta^1\theta^2)\theta^1 + f_2(t + \theta^1\theta^2)\theta^2 + f_{12}(t + \theta^1\theta^2)\theta^1\theta^2. \end{aligned}$$

And so we must only make sense of $f_I(t + \theta^1\theta^2)$. The key is that we take a formal *Taylor series expansion*; the series of course terminates due to the nilpotence of the the odd coordinates:

$$f_I(t + \theta^1\theta^2) = f_I(t) + \theta^1\theta^2 f_I'(t).$$

It is easy to check that this in fact gives a homomorphism of superalgebras. For $g, h \in C^\infty(\mathbb{R})$, $(gh)^* = gh + \theta^1\theta^2(gh)' = g^*h^*$. Notice moreover that in order to determine the sheaf morphism, it is enough to specify the images of the global sections since the full sheaf map is determined by restrictions of the global coordinates, as we shall see in complete and detailed generality later. So in this example, in fact, the morphism induced by equations (4.1) is unique via the above construction.

This fact is indeed true in general. As we shall see, the Chart Theorem states that a morphism ϕ between superdomains is determined by images of local coordinates under the sheaf morphism ϕ^*. Its proof requires a preliminary result on polynomial approximations of smooth sections, which is of interest by itself.

Lemma 4.1.9 (Hadamard's lemma). *Let $U^{p|q}$ be a superdomain, $f \in \mathcal{O}(U^{p|q})$ and $x \in |U^{p|q}| = U$. Then for each $k \in \mathbb{N}$ there exists a polynomial $P_{k,x}$ of degree k in $t^1 - t^1(x), \dots, t^p - t^p(x)$, $\theta^1, \dots, \theta^q$, such that*

$$f - P_{k,x} \in I_x^{k+1}$$

where $I_x = \{f \in \mathcal{O}(U^{p|q}) \mid f(x) = 0\}$.

Moreover if f and g are such that for a given $k \geq q$, $f - g \in I_x^{k+1}$ for each $x \in |U^{p|q}|$, then $f = g$.

Notice that I_x is the ideal generated by $\theta^1, \dots, \theta^q$ and by the maximal ideal of sections in $C^\infty(U)$ vanishing at x.

Proof. Consider $x \in U$ with coordinates $t_0^1, \dots, t_0^p$. Let $f = \sum_I f_I \theta^I \in C^\infty(U) \otimes \bigwedge^q$. For each given integer r, $f_I \in C^\infty(U)$ can be expanded using Taylor series:

$$f_I(t) = f_I(t_0) + \sum_{|\gamma| \leq r} \frac{1}{\gamma!} \left(\frac{\partial^{|\gamma|} f_I}{\partial t^\gamma} \right)_{t_0} (t - t_0)^\gamma + h_{I,r+1}(t)(t - t_0)^{r+1}.$$

Here $h_{I,r} \in C^\infty(U)$, $t = (t^1, \dots, t^p)$, and we are using a multi-index notation.

Notice that in this expression $f_I(t_0)$ is the value of the C^∞ function f_I in t_0 in the ordinary sense.

Define

$$P_{I,r}(t) = f_I(t_0) + \sum_{|\gamma| \leq r} \frac{1}{\gamma!} \left(\frac{\partial^{|\gamma|} f_I}{\partial t^\gamma} \right)_{t_0} (t - t_0)^\gamma,$$
$$R_{I,r+1}(t) = h_{I,r+1}(t)(t - t_0)^{r+1}.$$

Clearly $P_{I,r}$ is a polynomial in $t^1 - t^1(x), \dots, t^p - t^p(x)$ of degree r, while $R_{I,r+1} \in I_x^{r+1}$. Now define

$$P_{k,x}(t, \theta) = \sum_I P_{I,k-|I|}(t) \theta^I.$$

One can readily check that this satisfies the requirements of the statement.

Now we turn to the second part of the statement. Let $h = f - g \in I_x^{k+1}$ for all $x \in |U^{p|q}|$ with $k \geq q$. We want to show that $h = 0$. Since $\theta^I = 0$ for $|I| > q$ for all x, it follows that $h = \sum_{I,|I| \leq q} h_I \theta^I$, where $h_I \in I_x^r$ for some $r > 0$. By definition, $h_I(x) = 0$ for all $x \in U$, hence h_I is identically zero and we have $h = 0$. □

Remark 4.1.10. Before turning to the proof of the Chart Theorem we stress that the above lemma has a number of consequences that deserve attention, and that will be discussed more deeply in Section 4.3.

Theorem 4.1.11 (Chart Theorem). *Let $U \subset \mathbb{R}^{p|q}$ and $V \subset \mathbb{R}^{m|n}$ be open superdomains (with canonical charts). There is a bijection between*

(i) *the set of morphisms $\phi\colon V \to U$ and*

(ii) *the set of systems of p even functions t^{i*} and q odd functions θ^{j*} in $\mathcal{O}(V)$ such that $(t^{1*}(m), \dots, t^{p*}(m)) \in |U|$ for all $m \in |V|$.*

Proof. It is clear that if we have a morphism ϕ we can uniquely associate to it a set of p even and q odd functions in $\mathcal{O}(V)$. In fact, we take $t^{i*} := \phi^*(t^i)$, $\theta^{j*} := \phi^*(\theta^j)$, where $\{t^i, \theta^j\}$ is the canonical chart in $\mathcal{O}(U)$.

Assume now that (ii) holds. We denote by $\{t^i,\ \theta^j\}$ and $\{x^r,\ \xi^s\}$ coordinates on U and V, respectively. It is clear that we have a continuous map $|\phi|\colon |V| \to |U|$. We now need to show that there exists a unique superalgebra morphism $\phi^*\colon C^\infty(|U|) \otimes \bigwedge^p \to C^\infty(|V|) \otimes \bigwedge^n$ such that $\phi^*(t^i) = t^{i*}$ and $\phi^*(\theta^j) = \theta^{j*}$. Let us start with the existence of such ϕ^*. It is enough to show that there exists a superalgebra morphism $\phi^*\colon C^\infty(|U|) \to C^\infty(|V|) \otimes \bigwedge^n$, since we know the image of the polynomial odd generators θ^j. Let $t^{*i} = \widetilde{t^{*i}} + n^i$ with $\widetilde{t^{*i}} \in C^\infty(|V|)$, $n^i := \sum_{|I|>1} t_I^{*i} \xi^I$ and define the pullback, through a formal Taylor expansion, by

$$\phi^*(f) := \sum_\gamma \frac{1}{\gamma!} \frac{\partial^\gamma f}{\partial t^\gamma}\bigg|_{\widetilde{t^*}} n^\gamma, \tag{4.2}$$

where we are using a multi-index notation as in Lemma 4.1.9 and by $|_{\widetilde{t^*}}$ we mean to substitute any instance of t_i with $\widetilde{t_i^*}$.

We show that it is a superalgebra morphism. Indeed,

$$\begin{aligned}
\phi^*(f \cdot g) &= \sum_\gamma \frac{1}{\gamma!} \frac{\partial^\gamma (f \cdot g)}{\partial t^\gamma}\bigg|_{\widetilde{t^*}} n^\gamma \\
&= \sum_{\alpha<\gamma} \frac{1}{\gamma!} \binom{\gamma}{\alpha} \frac{\partial^\alpha f}{\partial t^\alpha}\bigg|_{\widetilde{t^*}} \frac{\partial^{\gamma-\alpha} g}{\partial t^{\gamma-\alpha}}\bigg|_{\widetilde{t^*}} n^\gamma \\
&= \sum_{\alpha,\beta} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial t^\alpha}\bigg|_{\widetilde{t^*}} n^\alpha \frac{1}{\beta!} \frac{\partial^\beta g}{\partial t^\beta}\bigg|_{\widetilde{t^*}} n^\beta .
\end{aligned}$$

We now come to uniqueness. Suppose that ϕ_1^* and ϕ_2^* are two morphisms $C^\infty(|U|) \otimes \bigwedge^q \to C^\infty(|V|) \otimes \bigwedge^n$ such that $\phi_1^*(t^i) = \phi_2^*(t^i) = t^{i*}$, $\phi_1^*(\theta^j) = \phi_2^*(\theta^j) = \theta^{j*}$. They clearly coincide on polynomial sections. If $x \in |U|$, $f \in C^\infty(|U|) \otimes \bigwedge^q$, due to Lemma 4.1.9, we can write, for $k > q$, $f = P_{k,x} + h$ with $h \in I_x^{k+1}$. Hence $(\phi_1^* - \phi_2^*)(P_{k,x} + h) = (\phi_1^* - \phi_2^*)(h)$. It is easily checked that $\phi_i^*(I_x^k) \subseteq I_y^k$ with y such that $|\phi|(y) = x$, so that the result follows in view of Lemma 4.1.9.

Given the superalgebra morphism $\phi^*\colon C^\infty(|U|)\otimes\bigwedge^q \to C^\infty(|V|)\otimes\bigwedge^n$ and an open subset $|W|\subseteq|U|$, reasoning as before we can define a morphism $\phi^*_W\colon C^\infty(|W|)\otimes\bigwedge^q \to C^\infty(|\phi|^{-1}(|W|))\otimes\bigwedge^n$ by

$$t^i|_W \to t^{i*}|_{|\phi|^{-1}(|W|)}, \quad \theta^j|_W \to \theta^{j*}|_{|\phi|^{-1}(|W|)}.$$

In this way it is easy to check that we have a sheaf morphism. We leave to the reader the easy check of all details. □

Remark 4.1.12. Equation (4.2) in the proof of the previous proposition gives the recipe for calculating the pullback of a generic section from the pullback of the super coordinates.

Notice also that because the expansion (4.2) involves an arbitrary number of derivatives, there is no straightforward way to make sense of C^k-supermanifolds.

4.2 The category of supermanifolds

In this section we introduce the category of supermanifolds. A supermanifold is essentially a super ringed space locally modelled by superdomains. Since, as we have seen in the previous section, smooth superdomains are superspaces, we have that also supermanifolds are superspaces.

We also define the important technical notion of partition of unity in the super context, which we shall use in many proofs, and we discuss the reduced manifold associated with a supermanifold.

Definition 4.2.1. A superspace $M = (|M|, \mathcal{O}_M)$ is called a *supermanifold* if

(i) $|M|$ is a (locally compact) second countable Hausdorff topological space

(ii) for each $x \in |M|$ there exists an open neighborhood $U \ni x$ such that there exists an isomorphism

$$(U, \mathcal{O}_M|_U) \xrightarrow{\varphi_U} U^{m|n} \subset \mathbb{R}^{m|n}$$

for fixed (i.e., independent of x) $m|n$, where $U^{m|n}$ is a superdomain in $\mathbb{R}^{m|n}$.

A *morphism* between supermanifolds is a morphism between the corresponding superspaces. The pair $m|n$ is called the *superdimension* (or *dimension* for short) of M.

We denote by (smflds) the category of supermanifolds. Clearly each superdomain is a supermanifold. The pair (U, φ_U) is called a *superchart* (or *chart*) around $x \in U$.

In analogy with Lemma 4.1.6, we can prove the following lemma that enables us to give precise meaning to the evaluation of a section in the structure sheaf.

Lemma 4.2.2. *Let M be a smooth supermanifold, U be an open subset of $|M|$, and $f \in \mathcal{O}_M(U)$. There exists a unique real number $f(x)$ such that $f - f(x)$ is not invertible in any neighborhood $U \subseteq V$ containing x. Moreover if (V', t^i, θ^j) is a chart containing x and contained in V, and $f|_{V'} = f_0 + \sum_{|I|\geq 1} f_I\theta^I$, then $f(x) = f_0(x)$.*

The next lemma shows that a morphism of supermanifolds as super ringed spaces is automatically a morphism of superspaces, hence of supermanifolds. This result is especially important for applications.

Lemma 4.2.3. *Let M and N be supermanifolds, and let $\phi\colon M \to N$, $\phi := (|\phi|, \phi^*)$ be a super ringed spaces morphism. Then*

(i) $\phi^*(f)(x) = f(|\phi|(x))$ *for all* $f \in \mathcal{O}_N(U)$,

(ii) ϕ *defines a supermanifold morphism.*

Proof. Let us start with the proof of (i). Suppose that $\phi^*(f)(x) \neq f(|\phi|(x))$. If $g = f - f(|\phi|(x)) \in \mathcal{O}_N(U)$, this means that $\phi^*(g)(x) \neq 0$. g is not invertible in any neighbourhood of $|\phi|(x)$, however, since ϕ^* is a superalgebra morphism and $\phi^*(g)(x) \neq 0$, we have a contradiction.

In order to prove (ii), we only need to prove that ϕ^* is a local morphism in the sense that if $J_{|\phi|(x)}$ denotes the maximal ideal of the stalk $\mathcal{O}_{M,|\phi|(x)}$ then

$$\phi^*(J_{|\phi|(x)}) \subseteq J_x.$$

Since J_x identifies with the germs of sections that evaluated at x are zero, (ii) follows directly from (i). □

Remark 4.2.4. We can state the following more general version of the Chart Theorem that is proven in the same way as its local version 4.1.11.

Theorem 4.2.5 (Global Chart Theorem). *Let $U \subset \mathbb{R}^{p|q}$ be a superdomain and M a supermanifold. There is a one-to-one correspondence between the morphisms $M \to U$ and the set of $p+q$-uples of p even functions t^{i*} and q odd functions θ^{j*} on M such that $(t^{1*}(x), \dots, t^{p*}(x)) \in |U|$ for all $x \in |M|$.*

Proof. One way is clear, namely the fact that for each morphism $\psi\colon M \to U$ we have the family of sections detailed above. Suppose now we are given a family of sections t^{i*}, θ^{j*} on M such that $(t^{1*}(x), \dots, t^{p*}(x)) \in |U|$ for all $x \in |M|$. We want to define a morphism $\psi\colon M \to U$. For each point $x \in |M|$, if $(V_x, \mathcal{O}_M|_{V_x})$ is a chart around x, we have a morphism $\psi_x\colon V_x \to U$ which is uniquely determined by the Chart Theorem for superdomains. More precisely ψ_x is the morphism corresponding to the assignment $\psi^*(x^i) = t^{i*}|_{V_x}$, $\psi^*(\xi^j) = \theta^{j*}|_{V_x}$, where x^i, ξ^j are local coordinates for the chart V_x. Now, if f is a section in $\mathcal{O}(U)$, we can consider the pullbacks $\psi_x^*(f)$ for each $x \in |M|$. Clearly $\psi_x^*(f)|_{V_x \cap V_y} = \psi_y^*(f)|_{V_x \cap V_y}$ for each x and y in $|M|$. Due to the sheaf property there exists a unique section $\psi^*(f)$, hence we have defined uniquely our morphism ψ^*. □

Remark 4.2.6. Any section $f \in \mathcal{O}_M(U)$ can now be interpreted as a morphism $f : U \to \mathbb{R}^{1|1}$, thus recovering the intuitive meaning of sections as functions on open subsets of the supermanifold. In fact as one can readily see, by the Chart Theorem an even function $f : U \to \mathbb{R}^{1|0}$ corresponds to the choice of an even section in $\mathcal{O}_M(U)$, while an odd function $f : U \to \mathbb{R}^{0|1}$ corresponds to the choice of an odd section in $\mathcal{O}_M(U)$. For this reason we shall write $|f|$, for a section $f \in \mathcal{O}_M(U)$, meaning the topological function $|f|: |U| \to \mathbb{R}$. Clearly $|f|(x) = f(x)$ as in Definition 4.1.1. We shall also use $\tilde{f}$ to denote $|f|$ whenever there is ambiguity with the absolute value.

If $|U|$ is an open subset of $|M|$, the superspace $(|U|, \mathcal{O}_M|_U)$ is itself a supermanifold and it is called the *open submanifold* determined by $|U|$. We shall use the term "submanifold" instead of the more cumbersome, though more precise, "subsupermanifold". In particular if M and N are supermanifolds, U is an open submanifold of N, and $|\phi|: |M| \to |N|$ is a continuous map, then we denote by $|\phi|^{-1}(U)$ the open submanifold of M determined by $|\phi|^{-1}(|U|)$.

In the next sections we shall investigate the structure of a supermanifold at different levels: infinitesimal, local and global. In order to do this, we need some preliminary results. We start with a definition of *partition of unity*.

As in ordinary differential geometry, partitions of unity are very useful in passing from global to local problems and vice versa.

Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold and let f be a section over U open in $|M|$. Consider the set of points $x \in U$, for which there exists an open neighborhood $V \subseteq U$ such that $f|_V = 0$. This is an open set, whose complement is called the *support* of f, $\mathrm{supp}(f)$.

We also recall that if $|M|$ is a topological space and $\{U_i\}_{i\in I}$ is a cover of M, we say that $\{V_j\}_{j\in J}$ is a *refinement* of $\{U_i\}_{i\in I}$ if for each V_j there exists U_i such that $V_j \subseteq U_i$.The refinement is said to be *locally finite* if each $x \in |M|$ has a neighbourhood intersecting only a finite number of V_j.

Proposition 4.2.7. *Let $\{U_i\}_{i\in I}$ be an open covering of $|M|$ then there exists a family $\{g_j\}$ of sections such that*

(1) $g_j \in \mathcal{O}(M)_0$,

(2) $\{\mathrm{supp}\, g_j\}_{j\in J}$ *is a locally finite refinement of* $\{U_i\}_{i\in I}$ *and* $\mathrm{supp}\, g_j$ *is compact for each* j,

(3) $\sum g_j = 1$ *and* $\tilde{g}_j \geq 0$[1] *for each* j.

There exists also a family $\{h_i\}_{i\in I}$ such that

(i) $h_i \in \mathcal{O}(M)_0$,

(ii) $\mathrm{supp}\, h_i \subset U_i$ *for all* $i \in I$ *of* $\{U_i\}_{i\in I}$,

(iii) $\sum h_i = 1$ *and* $\tilde{h}_i \geq 0$ *for each* i.

Proof. If $\{U_i\}_{i\in I}$ is an open cover of $|M|$, then there exists a locally finite refinement $\{V_j\}_{j\in J}$ where each V_j is an open subset with compact closure (see [78]). Without loss of generality assume that both U_i and V_j are supercharts coverings. Hence for each V_j there exists a superchart $U_{i(j)}$ containing it such that $\overline{V_j} \subset U_{i(j)}$. For each j define

[1] See Remark 4.2.6 for the notation.

an even section $f_j \in \mathcal{O}_M(U_{i(j)})_0$ such that supp $f_j =$ supp $\widetilde{f_j} \subseteq \overline{V_j}$ and $\widetilde{f_j} > 0$ for each $x \in V_j$. Each f_j can be identified with a global section $f_j \in \mathcal{O}(M)$ by defining it to be zero in $\overline{U^c_{i(j)}}$.[2] By the local finiteness condition $f = \sum f_j$ is a well-defined section in $\mathcal{O}(M)_0$ such that $\tilde{f} > 0$. Hence f is invertible and $g_j := \frac{f_j}{f}$ is the required partition of unity.

The second part of the proposition is proved as follows. For each $i \in I$ let $J_i := \{j \in J \mid V_j \subseteq U_i\}$ and define $g'_i = \sum_{k \in J_i} g_k$. Due to local finiteness the sum is well defined and supp $g'_i \subseteq U_i$. The sections $h_i := \frac{g'_i}{\sum g'_i}$ do the job. □

Definition 4.2.8. Let M be a supermanifold. A family of global sections satisfying the properties (i)–(iii) in Proposition 4.2.7 is called a *partition of unity*. It is customary to use the same terminology also for a family satisfying (1)–(3) in Proposition 4.2.7, but we shall make a comment whenever we need this stricter requirement.

Remark 4.2.9. It is worth noticing that in the previous definition, in points (1)–(3), we require the sections forming a partition of unity to have compact support. Hence we are forced to take for them a different index set from the one used for the open cover. On the other hand, in points (i)–(iii) we can take the two index sets to be the same, but then we have to abandon the compact support requirement.

Corollary 4.2.10. *If U and V are open subsets of a supermanifold M such that $U \subseteq \bar{U} \subseteq V$, then there exists an even section $f \in \mathcal{O}(M)_0$ such that $0 \le \tilde{f}(x) \le 1$ for all $x \in |M|$ and*

$$f|_U = 1, \quad f|_{\bar{V}^c} = 0.$$

Proof. The proof is immediate in view of the previous proposition considering the open cover of M given by $V, \bar{U}^c$. □

To each supermanifold M, we can associate an ordinary manifold $\widetilde{M}$, whose underlying topological space is $|M|$, and a morphism $j : \widetilde{M} \to M$, which (as we shall see later) is a *closed embedding*.

Define the super ideal sheaf

$$J_M(U) := J_{\mathcal{O}_M(U)} := \{f \in \mathcal{O}_M(U) \mid f \text{ is nilpotent}\}.$$

A section $f \in \mathcal{O}_M(U)$ belongs to $J_M(U)$ if and only if $f(x) = 0$ for all $x \in U$. This can be proven easily and we leave it to the reader as an exercise.

Definition 4.2.11. Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold. We define $\widetilde{M} = (|M|, \mathcal{O}_M/J_M)$ to be the *reduced ordinary manifold* associated to M (for the definition of quotient of sheaves, we refer the reader to Chapter 2, Section 2.2).

As we can readily, see $\widetilde{M}$ is an ordinary manifold.

[2] If A is a subset in a topological space X, then A^c denotes its complement in X, $\bar{A}$ its closure in X.

Remark 4.2.12. Even if the structure sheaf of a smooth supermanifold is isomorphic to the sheaf of sections of an exterior bundle (see [7]), we cannot think of a supermanifold simply as an exterior bundle over an ordinary manifold. Morphisms between supermanifolds mix both even and odd coordinates and thus C^∞_M cannot be realized as a subsheaf of $\mathcal{O}_M$ for an open neighborhood U of a supermanifold M; it follows that there is no natural morphism $M \to \widetilde{M}$. The symmetries of interest in these extensions of classical manifolds are those which place even and odd on the same level. Such symmetries are called *supersymmetries* and are at the foundation of the physical supersymmetry theory which aims to treat bosons and fermions on the same footing.

Next we want to show that the quotient sheaf $\mathcal{O}_M/J_M$ possesses the property $(\mathcal{O}_M/J_M)(U) = \mathcal{O}_M(U)/J_M(U)$; in other words, we do not need to take any sheafification to obtain the quotient sheaf. (See Definition2.2.7 for the notion of sheafification.)

Proposition 4.2.13. *Let M be a supermanifold of dimension $p|q$ and let $U \subseteq |M|$ be an open subset. The assignment $U \to \mathcal{O}_M(U)/J_M(U)$ is a sheaf on $|M|$ locally isomorphic to $C^\infty(\mathbb{R}^p)$.*

Proof. The fact that $U \to \mathcal{O}_M(U)/J_M(U)$ is a presheaf on $|M|$ locally isomorphic to $C^\infty(\mathbb{R}^p)$ is immediate. Hence we only have to show the sheaf property.

We need to prove that if we have a family $f_\alpha \in \mathcal{O}_M(U_\alpha)/J_M(U_\alpha)$ such that $f_\alpha|_{U_\beta} = f_\beta|_{U_\alpha}$, then there exists a unique $f \in \mathcal{O}_M(\bigcup U_\alpha)/J_M(\bigcup U_\alpha)$ with $f|_{U_\alpha} = f_\alpha$. Let $\{g_\alpha\}$ be a partition of unity in $\mathcal{O}_M(U)$, subordinated to the open cover $\{U_\alpha\}$ of U. Notice that the product $g_\alpha f_\alpha$ is a well-defined element in $\mathcal{O}_M(U)/\mathcal{J}_M(U)$, and define $\hat{f}_\alpha := g_\alpha f_\alpha$. Clearly each $\hat{f}_\alpha$ is a smooth section over U whose support is contained in U_α. If we now define $f := \sum_\alpha \hat{f}_\alpha$, it is very easy to check that $f|_{U_\alpha} = f_\alpha$. □

We can now immediately define a natural morphism $j: \widetilde{M} \to M$ as follows. $|j|: |M| \to |M|$ is the identity. The pullback $j^*_U: \mathcal{O}_M(U) \to C^\infty_{\widetilde{M}}(U)$ is simply the projection map $\mathcal{O}_M(U) \to \mathcal{O}_M(U)/J_M(U)$. Notice that $j^* f$ is the (ordinary) differentiable function on $|M|$ whose value at each $x \in |M|$ is $f(x) \in \mathbb{R}$, i.e., $(j^* f)(x) = f(x)$.

The correspondence between supermanifolds and their associated reduced manifolds can be extended to the morphisms in the following way. Let $\phi := (|\phi|, \phi^*)$ be a morphism between the supermanifolds M and N. In order to define $\tilde{\phi}: \widetilde{M} \to \widetilde{N}$, notice that $\phi^*(J_N) \subseteq J_M$. Hence ϕ^* descends to a well-defined map between the quotient sheaves and we have the following proposition that we leave to the reader as an exercise.

Proposition 4.2.14. *The assignments $M \to \widetilde{M}$ and $\phi \to \tilde{\phi}$ define a functor*

$$\text{(smflds)} \to \text{(mflds)}.$$

The next question we want to tackle is how to build supermanifolds by gluing supermanifold structures on an open covering of a topological space.

Let $(U_i, \mathcal{O}_i)$ be supermanifolds such that $|M| = \bigcup_i U_i$ is a locally compact, Hausdorff, second countable topological space, in which every U_i is an open subset. We define $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Suppose that we have a family of morphisms

$$f_{ij} : (U_{ji}, \mathcal{O}_j|_{U_{ij}}) \to (U_{ij}, \mathcal{O}_i|_{U_{ij}})$$

such that

(1) f_{ij} is an isomorphism of ringed superspaces,

(2) $|f_{ij}| = \mathrm{id}_{U_{ij}}$.

Definition 4.2.15. Let the notation be as above. We say that a supermanifold $(|M|, \mathcal{O}_M)$ is obtained by *glueing* the supermanifolds $(U_i, \mathcal{O}_i)$, if for each i there exists a sheaf isomorphism

$$\phi_i : (U_i, \mathcal{O}_M|_{U_i}) \to (U_i, \mathcal{O}_i)$$

with $|\phi_i| = \mathrm{id}_{U_i}$ and such that (with some abuse of notation)

$$f_{ij} = \phi_i \phi_j^{-1}$$

on $U_i \cap U_j$. The f_{ij}'s are called *transition functions*.

We have the following proposition.

Proposition 4.2.16. *Let $(U_i, \mathcal{O}_i)$ be as above. The f_{ij}'s satisfy the* ***cocycle conditions***

(1) $f_{ii} = \mathrm{id}$ *on* $(U_i, \mathcal{O}_i)$,

(2) $f_{ij} f_{ji} = \mathrm{id}$ *on* $(U_{ij}, \mathcal{O}_i|_{U_{ij}})$,

(3) $f_{ij} f_{jk} f_{ki} = \mathrm{id}$ *on* $(U_{ijk}, \mathcal{O}_i|_{U_{ijk}})$,

if and only if there exists a unique glueing of the $(U_i, \mathcal{O}_i)$.

Proof. See Proposition 2.2.12. □

Notice that if M is a supermanifold and $\{U_\alpha, \varphi_\alpha\}$ is a superatlas, then M is isomorphic to the supermanifold obtained by glueing the U_α with transition functions $\varphi_\beta^{-1}\varphi_\alpha$.

Definition 4.2.17. Let M and N be supermanifolds of dimensions $p|q$ and $r|s$, respectively. We define the *product of the supermanifolds* M and N to be the super ringed space

$$M \times N = (|M| \times |N|, \mathcal{O}_{M\times N}),$$

where the sheaf $\mathcal{O}_{M\times N}$ is defined by $\mathcal{O}_{M\times N}(U \times V) = C^\infty(x,t)[\theta,\eta]$ for coordinate neighborhoods $U(= (x,\theta)) \subset |M|$, $V(= (t,\eta)) \subset |N|$. One must show that gluing conditions are satisfied, but this calculation mimics that in the ordinary category and is left to the reader. So $M \times N$ is a $(p+r)|(q+s)$-dimensional supermanifold with $\widetilde{M \times N} = \widetilde{M} \times \widetilde{N}$. As in the ordinary category, $\mathcal{O}_{M \times N} \neq \mathcal{O}_M \otimes \mathcal{O}_N$; instead we must take the completion of the tensor product to get an equality.

We shall return to the important concept of product of supermanifolds in Section 4.5.

4.3 Local and infinitesimal theory of supermanifolds

In Chapter 2, we have introduced the notion of *stalk* $\mathcal{O}_{M,x}$ associated to a sheaf $\mathcal{O}_M$ at a point $x \in |M|$. The elements in $\mathcal{O}_{M,x}$ are called *germs of sections* at x. Formally

$$\mathcal{O}_{M,x} = \varinjlim \mathcal{O}_M(U)$$

where the direct limit is taken over all open sets U containing x. As for the ordinary case, also for supermanifolds the stalk can be characterized as follows:

$$\mathcal{O}_{M,x} \cong \mathcal{O}(M)/\sim.$$

Here

$$f \sim g \text{ if there exists an open set } U \subseteq M \text{ such that } f|_U = g|_U.$$

as one can readily check from the definitions (see [29], p. 14 for more details). This characterization explains why the concept of a stalk at x is well suited for the study of those properties of supermanifolds depending on the behavior of sections on arbitrarily small neighborhoods of the point x.

If $[f]$ is a germ at x, it makes sense to evaluate $[f]$ at x in the same way as we evaluate the sections. We need also to define the natural morphism

$$\epsilon_x \colon \mathcal{O}(M) \to \mathcal{O}_{M,x}, \quad f \mapsto [f]_x.$$

The next lemma provides another useful characterization of the stalk at a point of a supermanifold.

Lemma 4.3.1. *If* $\mathfrak{m}_x = \{f \in \mathcal{O}(M) \mid$ *there exists an open neighbourhood* U *of* x *such that* $f_U = 0\}$, *then*

$$\mathcal{O}_{M,x} \simeq \mathcal{O}(M)/\mathfrak{m}_x.$$

Proof. Consider the morphism $\epsilon_x \colon \mathcal{O}(M) \to \mathcal{O}_{M,x}$ defined above. ϵ_x is surjective. In fact, if f is a representative of the germ $[f]_x$, we can think of it as a section in $\mathcal{O}_M(U)$, where U is an appropriate neighborhood of x. If V is an open subset such that $V \subseteq \overline{V} \subseteq U$ and $\phi \in \mathcal{O}(M)$ is a section such that $\operatorname{supp} \phi \subseteq U$ and $\phi|_V = 1$, then $\phi \cdot f$ is a global representative of $[f]_x$. It is clear that the kernel of ϵ_x consists of the elements of $\mathcal{O}(M)$ that are zero in a neighborhood of x, i.e., of $\mathfrak{m}_x$. □

In Section 4.1 we have discussed Hadamard's Lemma 4.1.9, which holds also in the context of germs of functions. We simply restate it, leaving the proof to the reader as a simple exercise.

Lemma 4.3.2 (Local Hadamard lemma). *Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold,* $\dim(M) = (m|n)$*, x a point in $|M|$, t^i, θ^j a local coordinate system around x. Let f and g be sections defined in a neighbourhood U of x.*

(1) *If $[f] \in \mathcal{O}_{M,x}$, then for each $k \in \mathbb{N}$ there exists a polynomial $P_{k,x}$ of degree k in $[t]^i - t^i(x)$, $[\theta]^j$ such that*

$$[f] - P_{k,x} \in J_x^{k+1}$$

where $J_x = \{[f] \in \mathcal{O}_{M,x} \mid [f](x) = 0\}$.

(2) *Moreover, if $[f]$ and $[g]$ are germs at x such that, for a given $k \geq n$, $[f]-[g] \in J_y^{k+1}$ for each y in a neighbourhood of x, then $[f] = [g]$.*

$\mathcal{O}_{M,x}$ is a local ring, that is, $\mathcal{O}_{M,x}$ contains a unique homogeneous maximal ideal J_x. Now we want to clarify the structure of J_x.

Proposition 4.3.3. *Let $[f] \in \mathcal{O}_{M,x}$ and* $\dim(M) = (m|n)$.

(1) *If $[f](x) = 0$, then $[f] \in ([t^i - t^i(x)], [\theta^j]) \subset \mathcal{O}_{M,x}$. In particular we have*

$$J_x = ([t^i - t^i(x)], [\theta^j]) \subset \mathcal{O}_{M,x}, \quad \mathcal{O}_{M,x} = \mathbb{R} \oplus J_x.$$

Hence $\mathcal{O}_{M,x}$ is a local superalgebra with maximal ideal $m_x = J_x$.

(2) *If $f \in \mathcal{O}(M)$, $k > n$ and $[f]_x \in J_x^k$ for all x in an open set U, then $f|_U = 0$.*

Proof. This follows immediately from Lemma 4.3.2. □

At this point one may be tempted to expand in formal power series the germs at a point x. This is fact makes sense, provided we exert some care (as one should also do for the ordinary case). We are going to come back to this point in Remark 4.3.12.

Remark 4.3.4. This gives another, and independent, proof of the fact that each superdomain is a superspace. Indeed the lemma essentially establishes that each stalk $\mathcal{O}_{U^{p|q},x}$ has a unique maximal ideal given by the sections whose value at x is zero. More precisely we have the decomposition

$$\mathcal{O}_{U^{p|q},x} = \mathbb{R} \oplus J_x$$

with $J_x = ([t^i - t^i(x)], [\theta^j]) \subset \mathcal{O}_{U^{p|q},x}$, where J_x is the ideal of germs that are zero at x.

This allows us to characterize the value of a section $f \in \mathcal{O}_{U^{p|q}}(V)$ at a point $x \in V$ as the image of f under the natural morphism

$$\mathcal{O}_{U^{p|q}}(V) \to \mathcal{O}_{U^{p|q},x} \to \mathcal{O}_{U^{p|q},x}/J_x \cong \mathbb{R},$$

where J_x denotes the maximal ideal in the stalk $\mathcal{O}_{U^{p|q},x}$.

The next results characterize the ideals of finite codimensions in the stalks. We shall use them in Section 4.7.

Proposition 4.3.5. *Each ideal I of finite codimension of a stalk $\mathcal{O}_{M,x}$ contains an ideal of the form J_x^k for some $k \in \mathbb{N}$.*

Proof. First let us notice that, due to Proposition 4.3.3, J_x is a finitely generated $\mathcal{O}_{M,x}$-module. Consider then the chain of ideals

$$\mathcal{O}_{M,x} \supseteq J_x + I \supseteq J_x^2 + I \supseteq \cdots .$$

Since I has finite codimension and $J_x^k \supsetneq J_x^{k+1}$, we have that there exists k such that

$$J_x^k + I = J_x^{k+1} + I.$$

Hence

$$J_x^k \subseteq J_x^{k+1} + I.$$

So we have

$$J_x^k = (J_x^{k+1} + I) \cap J_x^k = J_x^{k+1} + I \cap J_x^k.$$

Due to Nakayama's lemma (see Appendix B) we have $I \cap J_x^k = J_x^k$. □

We are ready to define tangent spaces. Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold.

Definition 4.3.6. We define a *tangent vector* at $x \in |M|$ to be a (super) derivation of the stalk $\mathcal{O}_{M,x}$, i.e., a linear map

$$v\colon \mathcal{O}_{M,x} \to \mathbb{R}$$

such that

$$v(f \cdot g) = v(f)g(x) + (-1)^{|v||f|} f(x)v(g).$$

Remark 4.3.7. Notice that the sign $(-1)^{|v||f|} f$ is immaterial. In fact the sign appears only when f and v are odd, but in this case $f(x) = 0$.

Definition 4.3.8. The super vector space of all tangent vectors at a point $x \in |M|$ is called the *super tangent space* at x and is denoted by $T_x(M)$.

Any supermanifold morphism $\phi\colon M \to N$ induces a stalk morphism

$$\phi_x^*\colon \mathcal{O}_{N,|\phi|(x)} \to \mathcal{O}_{M,x},$$

which in turn defines a linear morphism of the tangent spaces

$$(\mathrm{d}\phi)_x\colon T_xM \to T_{\phi(x)}N, \quad v \mapsto v \circ \phi_x^*.$$

The linear map $(\mathrm{d}\phi)_x$ is called the *differential* of ϕ at x.

The differential $(\mathrm{d}\phi)_x$ is an even linear map of super vector spaces and for this reason, once we fix homogeneous bases for such vector spaces, it corresponds to a diagonal block matrix; in other words, it will not contain much information about the behaviour of the odd variables. For example, if ϕ is the morphism discussed in Example 4.1.8, one readily checks that $(\mathrm{d}\phi)_x$ is the identity for all x. In order to better study infinitesimally the odd directions, we need the concept of *Jacobian* that we shall introduce in the next section and fully discuss in Chapter 5.

In the next two propositions we provide very useful characterizations of the tangent space to a supermanifold.

Proposition 4.3.9. *Let $x \in |M|$. Suppose that $v \in T_x(M)$ and t^i, θ^j are a super coordinate system around x. Then:*

(1) *v is completely determined by $v([t^i]), v([\theta^j])$.*
(2) *Let $\{\frac{\partial}{\partial t^i}\big|_x\}_{i=1}^m$, $\{\frac{\partial}{\partial \theta^j}\big|_x\}_{j=1}^n$ be the derivations*

$$\frac{\partial}{\partial t^i}\Big|_x([t^k]) = \delta_{ik}, \quad \frac{\partial}{\partial t^i}\Big|_x([\theta^j]) = 0, \quad \frac{\partial}{\partial \theta^j}\Big|_x[t^k] = 0, \quad \frac{\partial}{\partial \theta^j}\Big|_x[\theta^l] = \delta_{jl}.$$

Then they form a basis of T_xM, hence $\dim M = \dim T_xM$.

Proof. Let g be a germ in $\mathcal{O}_{M,x}$. Notice that if g is a constant germ or it is in J_x^2 then $v(g) = 0$. Due to Proposition 4.3.3, we can write

$$g = \sum_I \Big(g_I(x) + \sum_i [t^i - t^i(x)]g_{i,I} + h_I(t^i)[t^i - t^i(x)]^2\Big)[\theta^I]$$

with $g_{i,I} \in \mathbb{R}$ and $h_I(t^i)[t^i - t^i(x)]^2 \in J_x^2$. Hence, with a mild abuse of notation, we have $v(g) = \sum g_{i,0}v([t^i]) + g_i(x)v([\theta^i])$. Hence if we consider the difference

$$V := v - \sum v([t^i])\frac{\partial}{\partial t^i}\Big|_x + v([\theta^j])\frac{\partial}{\partial \theta^j}\Big|_x,$$

it is immediate that V is a derivation that is zero on all the polynomials in $[t^i]$ and $[\theta^j]$. Since $v(J_x^k) \subseteq J_x^{k-1}$ and due to Proposition 4.3.3 we can conclude that $v = \sum v([t^i])\frac{\partial}{\partial t^i}\big|_x + v([\theta^j])\frac{\partial}{\partial \theta^j}\big|_x$. □

Let $x \in |M|$. Define the super vector space

$$\mathrm{Der}_x(\mathcal{O}(M), \mathbb{R}) = \{v\colon \mathcal{O}(M) \to \mathbb{R} \mid v(fg) = v(f)g(x) + (-1)^{|v||f|}f(x)v(g)\}.$$

Proposition 4.3.10. *The linear morphism*

$$\alpha\colon T_xM \to \mathrm{Der}_x(\mathcal{O}(M), \mathbb{R}), \quad v \to v \circ \epsilon_x,$$

is an isomorphism, where $\epsilon_x\colon \mathcal{O}(M) \to \mathcal{O}_{M,x}$ is the natural map.

Proof. Since ϵ_x is surjective, the map α is injective. To show that it is surjective, let us take $w \in \mathrm{Der}_x(\mathcal{O}(M), \mathbb{R})$. We want to show that $w(g) = 0$ for all $g \in \mathfrak{m}_x$, so that w induces a derivation $v\colon \mathcal{O}(M)/\mathfrak{m}_x \cong \mathcal{O}_{M,x} \to \mathbb{R}$, with $\alpha(v) = w$. So let us take $g \in \mathfrak{m}_x$. By definition of $\mathfrak{m}_x$, there exists an open subset $U \subseteq |M|$ such that $g|_U = 0$. We can find a section $f \in \mathcal{O}(M)$ such that $\operatorname{supp} f \subseteq U$ and $f(x) = 1$, by Corollary 4.2.10. So we have

$$0 = w(f \cdot g) = w(f)g(x) + f(x)w(g) = w(g).$$

This shows that w descends to a derivation v of the stalk $\mathcal{O}_{M,x}$ and that $\alpha(v) = w$. □

Observation 4.3.11. From the Propositions 4.3.9, 4.3.10 we immediately have the following facts:

(1) For any tangent vector $v \in T_x M$ and any neighbourhood U of x, there exists a unique derivation that we still denote by $v\colon \mathcal{O}_M(U) \to \mathbb{R}$.

(2) If (t^i, θ^j) are local coordinates in U, any derivation $v\colon \mathcal{O}_M(U) \to \mathbb{R}$ is determined once we know $v(t^i)$ and $v(\theta^j)$.

Remark 4.3.12. Consider the filtration

$$\mathcal{O}_{M,x} \supset J_x \supset J_x^2 \supset \cdots .$$

Elements of J_x^r will be denoted by $O(r)$. Due to the Proposition 4.3.3, J_x^r is generated as an ideal by the products

$$[(t^1 - t^1(x))]^{i_1} \dots [(t^m - t^m(x))]^{i_m} [\theta^{j_1}] \dots [\theta^{j_s}]$$

where $i_1 + \cdots + i_m + s = r$. Proceeding inductively, every $[f] \in \mathcal{O}_{M,x}$ can be written as

$$\begin{aligned} f &= f(x) + O(1) \\ &= f(x) + \sum_{1 < i_1 + \cdots + i_m + s \le r} a_{i_1 \dots i_m, j_1, \dots, j_s} [(t^1 - t^1(x))]^{i_1} \dots \\ &\qquad \dots [(t^m - t^m(x))]^{i_m} [\theta^{j_1}] \dots [\theta^{j_s}] + O(r+1) \end{aligned}$$

with $a_{i_1 \dots i_m, j_1, \dots, j_s} = \frac{1}{i_1! \dots i_m!} \frac{\partial^s}{\partial \theta^{j_1} \dots \partial \theta^{j_s}} \frac{\partial^{i_1 \dots i_m} f}{\partial (t^1)^{i_1} \dots \partial (t^m)^{i_m}}$.

One should be aware that there are germs in the intersection $\bigcap_k J_x^k$. These are the germs of sections whose germ is in $F_x \otimes \Lambda$ where F_x is the ideal of germs of flat functions at x, i.e., those functions with zero derivatives at all orders. Of course we cannot obtain any information about flat functions with power series expansion.

4.4 Vector fields and differential operators

Many concepts and results from ordinary differential geometry extend naturally to the category of supermanifolds. If we keep the categorical language we have developed, there is hardly any difference in fundamental differential geometry between the ordinary and the super categories. For example, we can define the super tangent bundle on M, where we find super extensions of the constant rank mapping theorem and the local and global Frobenius theorem, which we will prove in the following chapters.

Definition 4.4.1. A *vector field* V on a supermanifold M is an $\mathbb{R}$-linear derivation of $\mathcal{O}_M$, i.e., it is a family of super derivations $V_U : \mathcal{O}_M(U) \to \mathcal{O}_M(U)$ that is compatible with restrictions.

In a similar way, we can define vector fields over open submanifolds U of M.

Observation 4.4.2. When the supermanifold M is a superdomain, one can easily check that the vector fields on M are in one-to-one correspondence with the derivations $\mathcal{O}(M) \to \mathcal{O}(M)$. Moreover, one could also check, using partitions of unity, that this is true for generic smooth supermanifolds.

Definition 4.4.3. We define the *tangent bundle* Vec_M of the supermanifold M to be the sheaf consisting of all vector fields on M. This sheaf associates to each open set U in $|M|$ the vector superspace consisting of all derivations of $\mathcal{O}_M|_U$.

The sheaf Vec_M is actually locally free as a sheaf over $\mathcal{O}_M$, which we establish with the following lemma. In other words we have that there exists an open cover of $|M|$ such that $\mathrm{Vec}_M|_U \cong \mathcal{O}_M|_U^{p|q}$ for each U in the cover for suitable p and q. The lemma also helps us to understand the local structure of a vector field.

Lemma 4.4.4. *Let (t^i, θ^j) be coordinates on some open submanifold $U \subset \mathbb{R}^{p|q}$. Then the $\mathcal{O}_U$-module of $\mathbb{R}$-linear derivations of $\mathcal{O}_U$ is a rank $p|q$ free sheaf over $\mathcal{O}_U$ with basis $\{\partial/\partial t^i, \partial/\partial\theta^j\}$ where $\partial/\partial t^i$, $\partial/\partial\theta^j$ are the vector fields defined in U as*

$$\frac{\partial}{\partial t^i}(f_I(t)\theta^I) = \frac{\partial f_I(t)}{\partial t^i}\theta^I, \quad \frac{\partial}{\partial\theta^j}(f_I(t)\theta^j\theta^I) = f_I(t)\theta^I,$$

where $j \notin I$.

Proof. Assume that X is a vector field over the superchart (U, t^i, θ^j). It is a simple check that $X(I_x^p) \subseteq I_x^{p-1}$, where as usual $I_x = \{f \in \mathcal{O}_M(U) \mid f(x) = 0\}$. Define $\widehat{X} := \sum X(t^i)\frac{\partial}{\partial t^i} + \sum X(\theta^j)\frac{\partial}{\partial\theta^j}$. Clearly $\widehat{X}$ is also a superderivation. Consider the difference $D = X - \widehat{X}$. This is still a superderivation and, moreover, is zero on the polynomials in the t^i, θ^j. If $f \in \mathcal{O}_{\mathbb{R}^{p|q}}(U)$ then, due to Lemma 4.1.9, for each $x \in |U|$ and for $k > q + 1$, we can write $f = P_{k,x} + h$ with $h \in I_x^{k+1}$. Hence $D(f) = D(h) \in I_x^k$. Since x is arbitrary in U, we can conclude using Lemma 4.1.9 that $D = 0$. □

Definition 4.4.5. Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold. A rank $p|q$ locally free sheaf on M is called a *super vector bundle* on M.

Since $U \subset \mathbb{R}^{p|q}$ is the local model for any dimension $p|q$ supermanifold M, the lemma implies that Vec_M is a vector bundle of rank $p|q$. If V is a vector field on U, then in a coordinate neighborhood $U' \subset U$ with coordinates (t, θ), there exist functions f_i, g_j on U' so that V has the unique expression

$$V|_{U'} = \sum_{i=1}^{p} f_i(t,\theta)\frac{\partial}{\partial t^i} + \sum_{j=1}^{q} g_j(t,\theta)\frac{\partial}{\partial \theta^j}. \tag{4.3}$$

We have defined tangent vectors at x as $\mathbb{R}$-linear derivations $\mathcal{O}_x \to \mathbb{R}$ of the stalk at x; we may also think of a tangent vector $v \in T_x(M)$ as a (not uniquely determined) vector field on U, a neighborhood of x, composed with evaluation at x. If the open subset U from Definition 4.4.1 is a coordinate neighborhood around x, the vector v takes the expression

$$v = \sum a_i \frac{\partial}{\partial t^i}\bigg|_x + b_j \frac{\partial}{\partial \theta^j}\bigg|_x$$

for $a_i, b_j \in \mathbb{R}$, where

$$\frac{\partial}{\partial t^i}\bigg|_x := \mathrm{ev}_x \circ \frac{\partial}{\partial t^i}, \qquad \frac{\partial}{\partial \theta^j}\bigg|_x := \mathrm{ev}_x \circ \frac{\partial}{\partial \theta^j}$$

correspond to the tangent vectors we defined in Proposition 4.3.9. Notice that, contrary to the ordinary case, a vector field V on U is not determined by the family of tangent vectors: $\mathrm{ev}_x \circ V$.

For M and N supermanifolds, we can extend a vector on M to a $\mathcal{O}_N$-linear derivation on $M \times N$, and likewise we may trivially treat any vector field on M as a vector field on $M \times N$. We will call these extensions *extended vectors* and *extended vector fields*, respectively.

Definition 4.4.6. Let v be a tangent vector of M at m and $U_m \subset |M|$ an open neighborhood of m. We view v as a derivation $\mathcal{O}_M(U_m) \to \mathbb{R}$ and identify $\mathcal{O}_N$ with $\mathcal{O}_{\mathbb{R}\times N}$. Then v extends uniquely to a *$\mathcal{O}_N$-linear derivation*

$$\begin{array}{ccc} v_N : \mathcal{O}_{M\times N}(U_m \times V) & \longrightarrow & \mathcal{O}_N(V) \\ & \searrow \quad \swarrow & \\ & \mathcal{O}_{\mathbb{R}\times N}(\mathbb{R}\times V) & \end{array}$$

for any open $V \subset |N|$ (this is easily seen locally by using coordinates, and then by patching using local uniqueness) so that

$$v_N(a \otimes b) = v(a)b$$

where a and b are local functions of M and N, respectively.

One may similarly "extend" vector fields: let V be a vector field on M, and denote by $\mathbb{1}$ the identity operator on the sheaf of the supermanifold. Then we extend V to a derivation $(V \otimes \mathbb{1})$ on $M \times N$ by forcing V to act trivially on N. If (t, θ) and (x, ξ) are local coordinates on M and N respectively, V has the coordinate expression as in (4.3). Then the extension $(V \otimes \mathbb{1})$ has the same coordinate expression on $M \times N$ described by coordinates (t, x, θ, ξ), i.e., it is identically zero on (x, ξ). Again the extension is made to be unique by patching and using local uniqueness.

The fact that we can define v_N just by describing $v_N(a \otimes b)$ will be fully justified in Section 4.5.

We now want to consider the effect of a derivation on a pullback.

Proposition 4.4.7 (Chain rule). *Let $U^{p|q}$ and $V^{m|n}$ be superdomains and denote collectively by u^a and v^b the supercoordinates over $U^{p|q}$ and $V^{m|n}$, respectively. If $\psi : U^{p|q} \to V^{m|n}$ is a morphism, we have*

$$\frac{\partial \psi^*(f)}{\partial u^a} = \sum_b \frac{\partial \psi^*(v^b)}{\partial u^a} \psi^* \left(\frac{\partial f}{\partial v^b} \right). \tag{4.4}$$

Proof. The proof makes use of Lemma 4.1.9 and goes along the same line as that of Lemma 4.4.4. Let us write $D(f) := \frac{\partial \psi^*(f)}{\partial u^a}$ and $D'(f) := \sum_b \frac{\partial \psi^*(v^b)}{\partial u^a} \psi^* \left(\frac{\partial f}{\partial v^b} \right)$ and consider the derivation $D - D'$. Clearly $D - D'$ annihilates the supercoordinates v^b and hence all polynomials in them. If f is a generic section in $\mathcal{O}(V^{m|n})$ and $x \in V$ then, in view of Lemma 4.1.9, there exists a polynomial $P_{k,x}$ with $k = n + 1$ and $h \in I_x^{n+2}$ such that $f = P_{k,x} + g$. Hence $(D - D')(f) = (D - D')(g)$. On the other hand it is easy to check that $(D - D')(g)$ belongs to I_x^{n+1}. Since x is arbitrary in V, we can conclude that $D = D'$. □

In the ordinary setting equation (4.4) is rewritten in matrix form in terms of the *Jacobian matrix* and we have that the composition of morphisms of ordinary manifolds translates into matrix multiplication of their associated Jacobians. In the super setting this requires some care due to the appearance of signs. Let us start with a definition of the *Jacobian supermatrix* or *Jacobian* for short.

Definition 4.4.8. Let $\psi : M \to N$ be a morphism of supermanifolds. Let $x \in |M|$ be a point with a local coordinate system t^i, θ^j and $\psi(x)$ with local coordinate system s^k, η^l. We define the *Jacobian supermatrix* associated with a morphism ψ in a neighbourhood of x as

$$J_\psi := \left((-1)^{(|v^b|+1)|u^a|} \frac{\partial \psi^*(v^b)}{\partial u^a} \right)$$

where $(u^a) = (t^i, \theta^j)$ and $(v^b) = (s^k, \eta^l)$. Explicitly we have

$$\begin{pmatrix} \frac{\partial \psi^*(s^k)}{\partial t^i} & -\frac{\partial \psi^*(s^k)}{\partial \theta^j} \\ \frac{\partial \psi^*(\eta^l)}{\partial t^i} & \frac{\partial \psi^*(\eta^l)}{\partial \theta^j} \end{pmatrix}.$$

Proposition 4.4.9. *Let $\phi\colon M \to N$, $\psi\colon N \to P$ be two supermanifold morphisms and let u^a, v^b, w^c be local coordinate systems around points m, $\phi(m)$ and $\psi(\phi(m))$. Then*

$$J_{\psi\circ\phi} = J_\psi \cdot J_\phi.$$

Proof. Let $\tau := \psi \circ \phi$. By Proposition 4.4.7 we have

$$\begin{aligned}\frac{\partial \tau^*(w^c)}{\partial u^a} &= \sum_b \frac{\partial \phi^*(v^b)}{\partial u^a} \phi^* \frac{\partial \psi^*(w^c)}{\partial v^b} \\ &= \sum_b (-1)^{|v^b|(|w^c|+|u^a|)+1+|w^c||u^a|} \phi^* \frac{\partial \psi^*(w^c)}{\partial v^b} \frac{\partial \phi^*(v^b)}{\partial u^a}.\end{aligned}$$

(Recall that the parity of $\frac{\partial f}{\partial x}$ is $|f| + |x|$.)

So we can rewrite the expression, with some abuse of notation, as $J_{\psi\circ\phi} := J_\psi \cdot J_\phi$. Reintroducing even and odd coordinates t^i, θ^j and s^k, η^l on U and V respectively, we get our result. □

We end this section by briefly discussing the superalgebra of differential operators on a supermanifold M.

Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold.

Definition 4.4.10. Let U be an open subset of $|M|$. We define the *differential operators of degree k in U* as the subset of $\underline{\mathrm{Hom}}(\mathcal{O}_M(U), \mathcal{O}_M(U))$ defined inductively as follows. The differential operators of degree zero $\mathrm{Diff}_0(U)$ are the elements of $\mathcal{O}_M(U)$ acting multiplicatively. The differential operators of degree k are

$$\mathrm{Diff}_k(U) := \{D \in \underline{\mathrm{Hom}}(\mathcal{O}_M(U), \mathcal{O}_M(U)) \mid [D, f] \in \mathrm{Diff}_{k-1}(U) \quad \text{for all } f \in \mathcal{O}_M(U)\},$$

where $[D, f]$ denotes the commutator in $\underline{\mathrm{Hom}}(\mathcal{O}_M(U), \mathcal{O}_M(U))$ (see Chapter 1 for its definition).

Clearly

$$\mathrm{Diff}_{k-1}(U) \subseteq \mathrm{Diff}_k(U)$$

and each $\mathrm{Diff}_k(U)$ inherits from $\underline{\mathrm{Hom}}(\mathcal{O}_M(U), \mathcal{O}_M(U))$ the obvious gradation.

We define the *differential operators over U* as the subalgebra

$$\mathrm{Diff}(U) := \bigcup_{k=0}^{\infty} \mathrm{Diff}_k(U).$$

of $\underline{\mathrm{Hom}}(\mathcal{O}_M(U), \mathcal{O}_M(U))$. In particular

$$\mathrm{Diff}_1(U) = \mathcal{O}_M(U) \oplus \mathrm{Vec}_M(U)$$

and, as one can easily check, if $D \in \mathrm{Diff}_1(U)$, then $D(1) \in \mathcal{O}_M(U)$ and $(D - D(1)) \in \mathrm{Vec}_M(U)$.

Proposition 4.4.11. *The assignment* $U \mapsto \mathrm{Diff}(U)$ *defines a sheaf on* $|M|$*. For each positive integer* k*,* $U \mapsto \mathrm{Diff}_k(U)$ *is a subsheaf. For each* $x \in |M|$*, there exists a chart* (U, h) *containing* x *such that*

$$\mathrm{Diff}_k(U) \simeq \bigoplus_{|I+\gamma|\leq k} \mathcal{O}(U) \frac{\partial^{|I|}}{\partial t^I} \frac{\partial^{|\gamma|}}{\partial \theta^\gamma}.$$

Proof. The proof goes along the same lines as the classical one and is based on Lemma 4.4.4 together with an induction argument. We leave this to the reader as an exercise. □

4.5 Global aspects of smooth supermanifolds

The purpose of this section is to show that a smooth supermanifold structure on a topological space is fully encoded by the superalgebra of the global sections of the structure sheaf of the supermanifold.

This is a technical result and we shall achieve it in several steps. We first need the notion of *Fréchet superalgebra* and *Fréchet supersheaf*, which are very natural generalizations of the ordinary corresponding notions and will be our technical device for reconstructing the supersheaf. We have provided in Appendix C a brief summary of the definitions and results concerning Fréchet spaces and superspaces that we will need in this section.

We then study the *maximal spectrum* of the superalgebra $\mathcal{O}(M)$ of the global sections, consisting of all the maximal ideals. Such ideals are in one-to-one correspondence with the points of the topological space $|M|$ underlying the supermanifold and we shall see how to define a topology on the maximal spectrum so that we get a homeomorphism with the topological space $|M|$.

Finally our crucial result, Proposition 4.5.9, builds the sections of the supersheaf $\mathcal{O}_M$ over an open set U as certain quotients of global sections, thus allowing us to retrieve the full sheaf $\mathcal{O}_M$, just by the knowledge of $\mathcal{O}(M)$.

This section is independent from the rest and may be skipped for a first reading, with the one warning that the result in Proposition 4.5.4 is going to be used implicitly in many places. The reader willing to take it for granted can very well go to the next section.

We start by recalling two results which are proven in Appendix C using Fréchet spaces terminology.

Proposition 4.5.1. *Let* M *be a supermanifold of dimension* $m|n$.

(1) $\mathcal{O}_M$ *is a Fréchet supersheaf with respect to the family of seminorms*

$$p_{K,D}(f) := \sup_{x \in K} |\widetilde{(D(f))}(x)|, \quad f \in \mathcal{O}_M(U),$$

where D is a differential operator and K a compact in U.

Furthermore, $\mathcal{O}_M$ is a sheaf of Fréchet superalgebras; in other words, $\mathcal{O}_M(U)$ is a Fréchet superalgebra for all U with respect to the submultiplicative seminorms

$$q_{\alpha,K_n^i} := 2^{\alpha+2n} \max_{|\gamma|\leq\alpha,|I|} (p_{K_n^i,\frac{\partial}{\partial t^\gamma \partial\theta^I}}),$$

where $\{K_n^i\}_{i,n\in\mathbb{N}}$ is a countable family of compact subsets covering U with each K_n^i contained in a superchart.

(2) *For all open coverings $\{U_i\}$ of U, $\mathcal{O}_M(U)$ has the coarsest topology that makes the restriction $\mathcal{O}_M(U) \to \mathcal{O}_M(U_i)$ continuous.*

Proposition 4.5.2. (1) *If D is a super differential operator on M, then for each U, $D\colon \mathcal{O}_M(U) \to \mathcal{O}_M(U)$ is continuous.*

(2) *If $\psi\colon M \to N$ is a supermanifold morphism then $\psi^*\colon \mathcal{O}(M) \to \mathcal{O}(N)$ is continuous.*

As a consequence we have an important proposition which allows to construct morphisms on cartesian products of supermanifolds in a simple way. Let us first recall few well-known facts about the *projective tensor topology*.

Given two locally convex topological vector spaces V and W, it is possible to endow the algebraic tensor product $V \otimes W$ with several, in general inequivalent, locally convex topologies. We do not want to enter into the details and we refer the reader to the original references [40], [39] by Grothendieck or to [73]. The important point is that if $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are two open subsets, then there exists a unique topology, called the *projective tensor topology* on $C^\infty(U) \otimes C^\infty(V)$ such that

$$C^\infty(U) \hat{\otimes} C^\infty(V) \simeq C^\infty(U \times V),$$

where $C^\infty(U)\hat{\otimes}C^\infty(V)$ denotes as usual the corresponding completion. Nevertheless it is important to stress that the precise nature of the projective topology is not essential to our arguments, and that the topology induced by the family of seminorms in our examination is equivalent to it. This is due to a nice property of the spaces of smooth functions called *nuclearity*. We refer again to [73] for further details on this topic.

If V_i and W_i (i=1,2) are locally convex topological vector spaces and $\phi_i = V_i \to W_i$ are continuous linear maps, then the tensor product $\phi_1 \otimes \phi_2$ uniquely extends to a continuous map $\phi_1\hat{\otimes}\phi_2\colon V_1\hat{\otimes}V_2 \to W_1\hat{\otimes}W_2$.

We are ready to give a more intrinsic definition of product of supermanifolds than the one we have encountered in Definition 4.2.17. As one can readily see, the two definitions are equivalent.

Definition 4.5.3. Let M and N be supermanifolds.

We define the *product of the supermanifolds M* and N to be the super ringed space

$$M \times N = (|M| \times |N|, \mathcal{O}_{M\times N}),$$

where the sheaf $\mathcal{O}_{M\times N}$ is defined as follows. For each rectangular open set $U \times V \subseteq M \times N$ define the superalgebra

$$\mathcal{O}_{M\times N}(U \times V) := \mathcal{O}_M(U)\hat{\otimes}\mathcal{O}_N(V).$$

Due to Proposition 2.2.11, since the rectangular sets are a base for the product topology, it extends to a sheaf over $|M| \times |N|$ that is clearly locally trivial.

Involving again Proposition 2.2.11 (but this time item (2)), a similar reasoning can be used for the morphisms.

The next proposition formalizes the important facts about the product of two supermanifolds.

Proposition 4.5.4. (1) *$\mathcal{O}(M) \otimes \mathcal{O}(N)$ is dense in $\mathcal{O}(M \times N)$ when they are endowed with the projective tensor topology.*

(2) *If $\phi_i : M_i \to N_i$, $i = 1, 2$ are supermanifold morphisms, then the pullback of the map $\phi_1 \times \phi_2 : M_1 \times M_2 \to N_1 \times N_2$ is given by $\phi_1^* \hat{\otimes} \phi_2^*$ which is in turn completely determined by $\phi_1^* \otimes \phi_2^*$.*

Notice that this proposition allows us to simplify dramatically the definition of the pullback morphism on the sheaf on the product of two supermanifolds. We have already used this in Definition 4.4.6 and we are going to use it again many times in the text, mostly without mention.

We now turn to the problem of how to recover the topological space $|M|$ of a supermanifold $(|M|, \mathcal{O}_M)$ by the superalgebra of its global sections. We start with a simple proposition.

Proposition 4.5.5. *Every $x \in |M|$ defines the superalgebra morphism $\mathrm{ev}_x : \mathcal{O}(M) \to \mathbb{R}$ given by evaluation at x, and $\ker \mathrm{ev}_x = \mathcal{J}_x$ is a maximal ideal in $\mathcal{O}(M)$, with*

$$\mathcal{J}_x := \{f \in \mathcal{O}(M) \mid f(x) = 0\}.$$

Proof. This is immediate since $\mathcal{O}(M)/\mathcal{J}_x$ is a field. □

Definition 4.5.6. Let A be a commutative real superalgebra. The *real spectrum* of A is defined as the set of all maximal ideals in A.

$$\mathrm{MaxSpec}_{\mathbb{R}}(A) := \{\mathcal{M} \subset A \text{ is a maximal ideal in } A\}.$$

Classically, if M is an ordinary manifold, we have a one-to-one correspondence between the points of M and the maximal spectrum of the algebra of smooth functions on M:

$$\mathrm{MaxSpec}_{\mathbb{R}}(C^\infty(M)) \leftrightarrow |M|, \quad \ker(\mathrm{ev}_x) \leftrightarrow x.$$

This is a non-trivial result, playing the role of the Hilbert nullstellensatz in the context of smooth supermanifolds and goes under the name of Milnor exercise (see [58] for more details).

Proposition 4.5.7 (Super Milnor exercise). *All maximal ideals in $\mathcal{O}(M)$ are of the form $\mathcal{J}_x$ for some $x \in |M|$.*

Proof. We have already noticed that each $\mathcal{J}_x$ is a maximal ideal in $\mathcal{O}(M)$. Conversely, let I be a maximal ideal in $\mathcal{O}(M)$ and denote $\tilde{I}$ the corresponding ideal in $C^\infty(\widetilde{M})$, i.e., $\tilde{I} := \{f \in C^\infty(\widetilde{M}) \mid \text{there exists } g \in I \text{ such that } |g| = f\}$. $\tilde{I}$ is a maximal ideal in $C^\infty(M)$. It cannot be the whole $C^\infty(M)$ since otherwise I would contain the unit. Due to the classical Milnor exercise (see [58]) we have that there exists $x \in |M|$ such that $\tilde{I} = \widetilde{\mathcal{J}}_x$, where $\widetilde{\mathcal{J}}_x$ denotes the ideal of smooth functions vanishing at x. Let $\mathcal{J}_x$ be the preimage of $\widetilde{\mathcal{J}}_x$. Clearly $I \subseteq \mathcal{J}_x$, hence due to maximality $I = \mathcal{J}_x$. □

From now on, due to the previous proposition, we can identify the real spectrum $\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M))$ with $\mathrm{Hom}(\mathcal{O}(M), \mathbb{R})$, hence throughout this section we shall identify a point x with a maximal ideal $\mathcal{J}_x$ and with the morphism ev_x.

We now want to give to $\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M))$ a topological space structure.

For each point $\bar{x} \in |M|$, for each $n \in \mathbb{N}$, each n-tuple of elements $f_1, \dots, f_n \in \mathcal{O}(M)$ and each real number ϵ, we define the subset

$$B_\epsilon(\mathrm{ev}_{\bar{x}}; f_1, \dots, f_n) := \{\mathrm{ev}_x \in \mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M)) \mid |f_i(x) - f_i(\bar{x})| < \epsilon \text{ for all } i\}.$$

of $\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M))$. As one can readily check, these subsets define a base for a topology on $\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M))$.

Proposition 4.5.8. *The map*

$$\psi : |M| \to \mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M)), \quad x \mapsto \mathrm{ev}_x,$$

is a homeomorphism.

Proof. This is a classical result; we nevertheless include it for completeness. The fact that this map is a bijection follows immediately from the previous proposition. To show that it is a homeomorphism, we need to show that it is open and the preimage of an open set is open. It is enough to perform this check on the open sets of the basis. Let $U = B_\epsilon(\mathrm{ev}_{\bar{x}}; f_1, \dots, f_n)$ be an open subset in $\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M))$; since each f_i is a smooth section, we have $\psi^{-1}(U)$ is open.

Let now V be an open subset in $|M|$. We want to show that $\psi(V)$ is open. Let $\mathrm{ev}_x \in \psi(V)$ (i.e., $x \in V$). Choose U to be an open subset of $|M|$ such that $U \subseteq \overline{U} \subseteq V$. Let $f \in \mathcal{O}(M)$ be a section such that $f|_U = 1$ and $f|_{\overline{V}^c} = 0$. It is clear that $B_{\frac{1}{2}}(\mathrm{ev}_x; f) \subseteq \psi(V)$, hence any point ev_x in $\psi(V)$ has an open neighbourhood entirely contained in $\psi(V)$. □

We now want to see how it is possible to reconstruct the whole sheaf $\mathcal{O}_M$ starting from the supercommutative algebra of global sections $\mathcal{O}(M)$.

We will use the *localization* technique, borrowed from the algebraic setting. We briefly recall a few facts about it, referring the reader to [2], Ch. 3, for all the details.

The idea is the following: we want to invert some elements in the superalgebra of sections of the structure sheaf of a supermanifold. Since our rings are not commutative but *supercommutative*, we must exert some care.

Consider an open set $U \subset |M|$ and define the subset of sections in $\mathcal{O}(M)$ that are invertible over U. More precisely, put

$$\mathcal{S}_U := \{s \in \mathcal{O}(M)_0 \mid s|_U(x) \neq 0 \text{ for all } x \in U\}.$$

The set $\mathcal{S}_U$ is a multiplicatively closed subset of $\mathcal{O}(M)_0$, i.e., $1 \in \mathcal{S}_U$, and it is closed under multiplication. Hence we can localize $\mathcal{O}(M)$ as an $\mathcal{O}(M)_0$-module with respect to $\mathcal{S}_U$. This amounts to defining the set

$$\mathcal{S}_U^{-1}\mathcal{O}(M) := (\mathcal{S}_U \times \mathcal{O}(M))/\sim,$$

where the equivalence relation is given by: $(s, f) \sim (s', f')$ if and only if there exists $s'' \in \mathcal{S}_U$ such that

$$s''(s'f - sf') = 0.$$

By construction $\mathcal{S}_U^{-1}\mathcal{O}(M)$ is an $\mathcal{O}(M)_0$-module, however $\mathcal{S}_U^{-1}\mathcal{O}(M)$ is also a superalgebra if we define addition and multiplication by

$$(s, f) + (s', f') := (ss', s'f + sf'), \quad (s, f) \cdot (s', f') := (ss', ff').$$

The next result is crucial for the reconstruction of the structure sheaf of the supermanifold from the superalgebra of its global sections.

Proposition 4.5.9. *The map*

$$\ell\colon \mathcal{S}_U^{-1}\mathcal{O}(M) \to \mathcal{O}_M(U), \quad (s, f) \mapsto \frac{f|_U}{s|_U},$$

is a superalgebra isomorphism.

Proof. Let us first prove that ℓ is injective. Suppose hence that $\frac{f|_U}{s|_U}$ is zero. This means that $f|_U = 0$. Let $s \in \mathcal{S}_U$ be such that $s|_{\overline{U}^c} = 0$, then $sf = 0$, and injectivity is proved.

We now come to surjectivity. Let $f \in \mathcal{O}_M(U)$. We want to determine $k \in \mathcal{O}(M)$ and $h \in \mathcal{S}_U$ such that $f = k|_U/h|_U$. Let $\{U_i\}$ be a collection of open sets with compact closure such that $\overline{U_i} \subseteq U_{i+1}$ and $\bigcup_i U_i = U$. Let $\{g_i\}$ be sections in $\mathcal{O}(M)$ such that $g_i|_{U_i} = 1$, $\operatorname{supp} g_i \subseteq U_{i+1}$ and $0 \leq |g_i| \leq 1$ (see Corollary 4.2.10).

Consider

$$k = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{g_i f}{1 + \hat{q}_i(g_i) + \hat{q}_i(g_i f)} \quad \text{and} \quad h = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{g_i}{1 + \hat{q}_i(g_i) + \hat{q}_i(g_i f)},$$

where

$$\hat{q}_i := \max\{q_{i1}, \ldots, q_{ii}\}.$$

(For the definition of the submultiplicative seminorms q_i's refer to Proposition 4.5.1.) We claim that they are both well-defined sections. Let us prove this fact, for example, for the section k. We prove that the series $\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{g_i f}{1+\hat{q}_i(g_i)+\hat{q}_i(g_i f)}$ converges by showing that it is Cauchy. Nevertheless this is immediate from

$$\hat{q}_u\left(\sum_{i=r}^{s} \frac{1}{2^i} \frac{g_i f}{1+\hat{q}_i(g_i)+\hat{q}_i(g_i f)}\right) \leq \sum_{i=r}^{s} \frac{1}{2^i} \frac{\hat{q}_u(g_i f)}{1+\hat{q}_i(g_i)+\hat{q}_i(g_i f)} \leq \sum_{i=r}^{s} \frac{1}{2^i}$$

for $r, s \geq u$. Hence the map is surjective. □

Corollary 4.5.10. *Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold. Then the superalgebra of global sections $\mathcal{O}(M)$ determines the sheaf $\mathcal{O}_M$:*

$$|M| \supset U \mapsto \mathcal{S}_U^{-1}\mathcal{O}(M) \cong \mathcal{O}_M(U).$$

Remark 4.5.11. (1) It is important to remark that, as it happens already in the ordinary case, $\mathcal{O}_M(U)$ is in general larger than $\mathcal{O}(M)|_U$, that is, there are sections on U that do not come as the restriction of global sections (e.g., $1/x \in \mathbb{C}^\infty(\mathbb{R}^\times)$ is not the restriction of any global section on $\mathbb{R}$).

(2) The superalgebra $\mathcal{O}(M)$ does not embed into its localization $\mathcal{S}_U^{-1}\mathcal{O}(M) \cong \mathcal{O}_M(U)$. This has nothing to do with the odd nilpotents, but it is a phenomenon we already observe at the ordinary level. Thanks to the partition of unity, we can have non-zero global sections which are zero on an open set.

4.6 The functor of points of supermanifolds

In the previous section we have shown that, starting from the superalgebra of global sections of a smooth supermanifold M, it is possible to reconstruct both the underlying topological space $|M|$, and the sheaf $\mathcal{O}_M$.

In this section we want to give, as a fundamental application of the results of the previous section, an effective way to compute the T-points of a supermanifold. In fact we will show that a morphism between the superalgebras of global sections of two supermanifolds M and N completely determines a morphism of the corresponding smooth supermanifolds, thus identifying the T-points of a supermanifold M with the morphisms between $\mathcal{O}(M)$ and $\mathcal{O}(T)$, which are in general much more manageable.

Proposition 4.6.1. *Let M and N be smooth supermanifolds. There is a bijective correspondence*

$$\mathrm{Hom}(M, N) \leftrightarrow \mathrm{Hom}(\mathcal{O}(N), \mathcal{O}(M)).$$

Proof. One direction is completely straightforward, so let us concentrate on the other one.

Suppose that $\psi\colon \mathcal{O}(N) \to \mathcal{O}(M)$ is a superalgebra morphism. We define a supermanifold morphism $(|\phi_\psi|, \phi_\psi^*)$ in the following way. The reduced map is

$$|\phi_\psi|\colon \operatorname{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M)) \to \operatorname{MaxSpec}_{\mathbb{R}}(\mathcal{O}(N)), \quad \mathrm{ev}_x \to \mathrm{ev}_x \circ \psi.$$

The sheaf morphism ϕ_ψ^* is defined, using Proposition 4.5.9, by

$$\phi_{\psi,U}^*\colon \mathcal{O}_N(U) \to \mathcal{O}_M(|\phi_\psi|^{-1}(U)), \quad (s, f) \mapsto (\psi(s), \psi(f)).$$

Let us now check that $(|\phi_\psi|, \phi^*)$ is really a supermanifold morphism .

The reduced map $|\phi_\psi|$ is well defined since $\mathrm{ev}_x \circ \psi$ is a superalgebra morphism. It remains to prove that it is continuous. Let

$$U = B_\epsilon(\mathrm{ev}_{\bar{x}}; f_1, \dots, f_n) = \{\mathrm{ev}_z \in \operatorname{MaxSpec}_{\mathbb{R}}(\mathcal{O}(N)) \mid |f_i(z) - f_i(\bar{x})| < \epsilon\}$$

be an open subset in $\operatorname{MaxSpec}_{\mathbb{R}}(\mathcal{O}(N))$. We have

$$\begin{aligned} |\phi_\psi|^{-1}(U) &= \{\mathrm{ev}_y \in \operatorname{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M)) \mid |\langle \mathrm{ev}_y \psi - \mathrm{ev}_{\bar{x}}, f_j\rangle| \le \epsilon\} \\ &= \{\mathrm{ev}_y \in \operatorname{MaxSpec}_{\mathbb{R}}(\mathcal{O}(M)) \mid |\langle \mathrm{ev}_y, \psi(f_j)\rangle - f_j(\bar{x})| \le \epsilon\}, \end{aligned}$$

which is open due to the smoothness of the sections $\psi(f_j)$.

The fact that ϕ_ψ^* defines a sheaf morphism is easy. □

Summarizing all of the results of this and the previous section, we can state the following theorem, which is one of the main results of this chapter.

Theorem 4.6.2. *The functor*

$$F\colon (\text{smflds}) \to (\text{salg})$$

that assigns to each supermanifold M the supercommutative algebra $\mathcal{O}(M)$ and to each morphism $(|\phi|, \phi^*)$ the superalgebra map ϕ_M^* is a full and faithful embedding.

Proof. The fact that F is a functor is straightforward. The bijectivity on the objects is a consequence of Proposition 4.5.8 or in any case is coming from the classical result. The fully faithfulness amounts to Proposition 4.6.1. □

One of the striking consequences of Theorem 4.6.2 is that morphisms of supermanifolds can be assigned at the level of global sections. In other words, a morphism $M \to N$ is determined by the knowledge of just the morphism induced on the superalgebras of global sections $\mathcal{O}(N) \to \mathcal{O}(M)$. We shall use this result many times in the remaining chapters. We are now going to see in the next examples how simple the description of the functor of points of a supermanifold becomes with the help of Theorem 4.6.2.

Examples 4.6.3. (1) *The super vector space* $\mathbb{R}^{m|n}$ *as superdomain*. As we have seen in Chapter 3, Section 3.4, if T is any supermanifold, we have that the T-points of the supermanifold $\mathbb{R}^{m|n}$ are given by

$$\mathbb{R}^{m|n}(T) = \mathrm{Hom}(T, \mathbb{R}^{m|n}) = \mathrm{Hom}(\mathcal{O}(\mathbb{R}^{m|n}), \mathcal{O}(T)).$$

This means that a supermanifold morphism $f: T \to \mathbb{R}^{m|n}$ corresponds to a superalgebra morphism $f^*: \mathcal{O}(\mathbb{R}^{m|n}) \to \mathcal{O}(T)$ and by Hadamard's lemma such a morphism is known once we know the image of the elements in the canonical chart: $f^*(t^i)$, $f^*(\theta^j)$. Vice versa, the Chart Theorem tells us that for any choice of m even elements and n odd elements in $\mathcal{O}(T)$ there is a unique corresponding morphism $T \to \mathbb{R}^{m|n}$. Hence

$$\begin{aligned}\mathbb{R}^{m|n}(T) &\cong \{(f^*(t^1), \dots, f^*(t^m),\ f^*(\theta^1), \dots, f^*(\theta^n)) \mid f: T \to \mathbb{R}^{m|n}\}\\ &= \mathcal{O}(T)_0^m \oplus \mathcal{O}(T)_1^n = (\mathcal{O}(T) \otimes \mathbb{R}^{m|n})_0,\end{aligned}$$

where the first $\mathbb{R}^{m|n}$ is the functor of points of the supermanifold $\mathbb{R}^{m|n}$, while the second $\mathbb{R}^{m|n}$ is the real super vector space of superdimension $m|n$, $\mathbb{R}^{m|n} = \mathbb{R}^m \oplus \mathbb{R}^n$. In other words, a T-point of the supermanifold $\mathbb{R}^{m|n}$ consists of an $m+n$-uple of m even and n odd global sections of $\mathcal{O}(T)$. We realize that the notation $\mathbb{R}^{m|n}$ is used here to denote three different objects: the superdomain, its functor of points and the real super vector space; however this abuse of notation is justified by the above identification, it is widely spread, and we shall make sure that the context clarifies at each time which one we are referring to.

(2) *The general linear supergroup*. Let T be a supermanifold. In Example 3.1.5, we have described the super vector space of matrices $\mathrm{M}_{m|n} \cong \mathbb{R}^{m^2+n^2|2mn}$. Reasoning as in (1), we have that $\mathrm{M}_{m|n}(T)$ can be identified with the endomorphisms of the $\mathcal{O}(T)_0$-module $\mathbb{R}^{m|n}(T) \cong (\mathcal{O}(T) \otimes \mathbb{R}^{m|n})_0$, by rearranging the m^2+n^2 even sections and the $2mn$ odd ones in a matrix form with diagonal block matrices with even entries and off diagonal matrices with odd entries:

$$\mathrm{M}_{m|n}(T) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\} = (\mathrm{M}_{m|n})_0 \otimes \mathcal{O}(T)_0 \oplus (\mathrm{M}_{m|n})_1 \otimes \mathcal{O}(T)_1.$$

Here $A = (a_{ij})$, $B = (\beta_{il})$, $C = (\gamma_{kj})$, $D = (d_{kl})$ with $a_{ij}, d_{kl} \in \mathcal{O}(T)_0$, and $\beta_{il}, \gamma_{kj} \in \mathcal{O}(T)_1$. $(\mathrm{M}_{m|n})_0$ and $(\mathrm{M}_{m|n})_1$ denote respectively the even and odd part of the super vector space of matrices $\mathrm{M}_{m|n}$ (again the same symbol $M_{m|n}$ has several different meanings).

Notice that in doing this, we reobtain $\mathrm{Mat}(A^{m|n})_0$ as discussed in Chapter 1, Section 1.4, with $A = \mathcal{O}(T)$.

Define $F(T)$ as the group of automorphisms of $\mathbb{R}^{m|n}(T)$. Clearly $F(T) \subset \mathrm{M}_{m|n}(T)$. We want to show that $F = \mathrm{GL}_{m|n}$, i.e., F is the functor of points of the *general linear supergroup*, the supermanifold defined in Chapter 3, Sections 3.1–3.2. In that example,

$\mathrm{GL}_{m|n}$ is defined as the open submanifold of the supermanifold $\mathrm{M}_{m|n}$, whose reduced space consists of diagonal block $m+n \times m+n$ real matrices with non-zero determinant:

$$|\,\mathrm{GL}_{m|n}\,| = \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \mid \det(P) \neq 0,\ \det(Q) \neq 0 \right\}.$$

By Proposition 4.5.9 we have $\mathcal{O}(\mathrm{GL}_{m|n}) = \mathcal{S}^{-1}_{|\mathrm{GL}_{m|n}|}\mathcal{O}(\mathrm{M}_{m|n})$.

Hence

$$\begin{aligned} \mathrm{GL}_{m|n}(T) &= \mathrm{Hom}(S^{-1}_{|GL_{m|n}|}\mathcal{O}(\mathrm{M}_{m|n}), \mathcal{O}(T)) \\ &\subset \mathrm{Hom}(\mathcal{O}(\mathrm{M}_{m|n}), \mathcal{O}(T)) = \mathrm{M}_{m|n}(T). \end{aligned}$$

The elements in $\mathrm{Hom}(\mathcal{S}^{-1}_{|\mathrm{GL}_{m|n}|}\mathcal{O}(\mathrm{M}_{m|n}), \mathcal{O}(T))$ correspond to $(m|n \times m|n)$-matrices with entries in $\mathcal{O}(T)$ whose diagonal blocks are invertible once reduced. We leave it to the reader as a simple exercise to check that those correspond to $(m|n \times m|n)$-matrices with entries in $\mathcal{O}(T)$ whose diagonal blocks are invertible (recall that $t + n \in \mathcal{O}(T)$, n nilpotent, is invertible if and only if t is invertible).

Notice that while in general, as we pointed out in Remark 4.5.11, $\mathcal{O}(M)$ does not embed into $\mathcal{S}_U^{-1}\mathcal{O}(M)$, it does in this special case, and this is because here $U = |\mathrm{GL}_{m|n}|$ is dense in $|\mathrm{M}_{m|n}|$.

Hence

$$\mathrm{GL}_{m|n}(T) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A, D \text{ invertible} \right\} \subset \mathrm{M}_{m|n}(T).$$

This is what we have defined as $\mathrm{GL}_{m|n}(A)$ in Definition 1.4.4 for $A = \mathcal{O}(T)$.

4.7 Distributions with finite support

This section generalizes the previous one and we include it since, in Kostant's original approach to supermanifolds, Sweedler duals play a central role. Finite support distributions play an important role also in classical differential geometry (see, for example, [26], [17]). We want, however, to remark that we shall not take this point of view in examining the theory of supergeometry, so this section is not going to be used in the sequel, except for those instances in which we relate our treatment with Kostant's original one.

Suppose that $v \in T_x M$. Then $\ker v$ contains J_x^2, which, due to Proposition 4.3.3, is an ideal of finite codimension in $\mathcal{O}_{M,x}$. Alternatively, using the characterization in Observation 4.3.10, a direct computation shows that $\ker v \supseteq \mathcal{J}_x^2$, where

$$\mathcal{J}_x := \{ f \in \mathcal{O}(M) \mid f(x) = 0 \}.$$

Using partitions of unity and Proposition 4.3.3, it is easy to prove that also $\mathcal{J}_x^p$ is a finite codimension ideal for each $x \in |M|$ and $p \in \mathbb{N}$.

Definition 4.7.1. Let A be a superalgebra and denote by A^* the corresponding algebraic dual. The *Sweedler dual* of A is defined as

$$A^\circ = \{X \mid X \in A^*, \ker X \text{ contains an ideal of finite codimension}\}.$$

It is well known that the importance of the Sweedler dual stems from the fact that it inherits a coalgebra structure from the algebra structure of $\mathcal{O}(M)$. Indeed if an algebra A is not finite-dimensional then the adjoint of the multiplication

$$\langle \Delta X, a \otimes b \rangle := \langle X, a \cdot b \rangle.$$

does not turn A^* into a co-algebra since

$$(A \otimes A)^* \supsetneq A^* \otimes A^*.$$

On the other hand, A° is a coalgebra with respect to the comultiplication Δ defined above and counit

$$\eta \colon A^\circ \to \mathbb{R}, \quad X \mapsto \langle X, 1 \rangle.$$

The proof essentially follows the classical one (see [72]) and we are going to return to this with more details in Chapter 7.

Following the classical notation, let $\mathcal{O}(M)'$ denote the topological dual of $\mathcal{O}(M)$, that is the set of continuous linear operators $\mathcal{O}(M) \to \mathbb{R}$. We shall call such operators *distributions* in analogy with the classical terminology.

Definition 4.7.2. Let ϕ be a distribution in $\mathcal{O}(M)'$. A point x in $|M|$ is said to be in the *support* of ϕ if for each neighborhood U of x there exists a section $f \in \mathcal{O}(M)$ supported in U such that $\langle \phi, f \rangle \neq 0$. If the support of ϕ consists of a finite number of points, we say that ϕ is a *finite support distribution*; if the support of ϕ consists of the point x, we say that ϕ is a *finite support distribution at* x or a *point supported distribution*.

The reader can easily check that the support of a distribution is a closed subset of $|M|$. As we are going to see presently, the finite supported distributions are identified with the Sweedler dual. Before this, we need the characterizations of the finite support distributions and the finite codimension ideals.

Proposition 4.7.3. (1) *Let ϕ be a finite support distribution at x and let (t^i, θ^j) be local coordinates around x. Then*

$$\phi = \sum a_{IJ} \frac{\partial}{\partial t^I}|_x \frac{\partial}{\partial \theta^J}|_x.$$

(2) *Let ϕ be a finite support distribution. Then $\phi = \sum_i \phi_{x_i}$, where ϕ_{x_i} is a finite support distribution at x_i.*

Proof (Sketch). (1) Let U be a domain where the local coordinates (t^i, θ^j) make sense and temporarily denote by $O_M(U)^\star$ the space of distributions with finite support. Since $\operatorname{supp}\phi \subset U$, we have that ϕ is completely determined by $\phi|_{\mathcal{O}_M(U)}$. Since U is a domain, $\mathcal{O}_M(U) \cong C^\infty(U) \otimes \wedge(\theta^j)$, hence $O_M(U)^\star \simeq C^\infty(U)^\star \otimes \wedge(\theta^j)^\star$ Since $\wedge(\theta^j)$ is finite-dimensional, its topological dual coincides with its dual as a vector space. Moreover, since by Schwarz's theorem (see [64], p. 150) we have that finitely supported distributions at x are identified with differential operators evaluated at x, we immediately obtain the result.

(2) Locally at each point of its support, ϕ can be written as a point supported distribution. Using a suitable partition of unity we obtain the global expression of ϕ as the sum of differential operators (that is, point supported distributions) at each point of the support. □

There is a topological interpretation of the Sweedler dual, providing its "concrete" realization. This is the content of the next proposition. Before this however we need a lemma characterizing ideals of finite codimension.

Lemma 4.7.4. *Let I be an ideal of finite codimension in $\mathcal{O}(M)$. Then there exists a finite set $x_1, \dots, x_n$ in $|M|$ and an n-tuple of integers $p_1, \dots, p_n$ such that*

$$\mathcal{J}_{x_1}^{p_1} \cap \dots \cap \mathcal{J}_{x_n}^{p_n} \subseteq I \subseteq \mathcal{J}_{x_1} \cap \dots \cap \mathcal{J}_{x_n}.$$

Proof. Let U be an open subset of $|M|$ and write $\mathcal{O}_M(\bar{U}) := \mathcal{O}(M)|_U$. In particular $C^\infty(\bar{U}) = C^\infty(M)|_U$ identifies with the restriction of smooth functions to the closed subset $\bar{U}$.

If J is a finite codimension ideal in $\mathcal{O}_M(\bar{U})$, let $\tilde{J} \subseteq C_M^\infty(\bar{U})$ denote the associated reduced ideal. Clearly $\tilde{J}$ is a finite codimension ideal in $C_M^\infty(\bar{U})$, and $\tilde{J} = C_M^\infty(\bar{U})$ if and only if $J = \mathcal{O}_M(\bar{U})$. If $f \in C_M^\infty(\bar{U})$, let $Z_f := f^{-1}(0)$, and let

$$Z_J := Z_{\tilde{J}} := \bigcap_{f \in J} Z_{\tilde{f}}.$$

It is well known (see, for example, [58]) that Z_J is a finite subset $\{x_1, \dots, x_n\}$ of $|M|$, that $Z_J = \emptyset$ if and only if $\tilde{J} = C_M^\infty(\bar{U})$, and that

$$\tilde{J} \subseteq \tilde{\mathcal{J}}_{Z_J} := \tilde{\mathcal{J}}_{x_1} \cap \dots \cap \tilde{\mathcal{J}}_{x_n}.$$

Taking $U = M$, it then follows that $J \subseteq \mathcal{J}_{x_1} \cap \dots \cap \mathcal{J}_{x_n}$.

We now claim that if V is an open subset of $|M|$ such that $Z_J \subseteq V$, and $f \in \mathcal{O}(M)$ is such that $f|_V = 0$, then $f \in I$. Indeed let U be another open subset of $|M|$ such that $Z_J \subseteq U \subseteq \bar{U} \subseteq V$, and denote by $\mathcal{O}_M(M \setminus U)$ the supersubalgebra of $\mathcal{O}(M)$ whose elements are restrictions of global sections to $\bar{U}^c$. Clearly the restriction $\rho\colon \mathcal{O}(M) \to \mathcal{O}_M(M \setminus U)$ is surjective, moreover $\rho(I)$ is a finite codimension ideal of $\mathcal{O}_M(M \setminus U)$ with $Z_J = \emptyset$. Then $\rho(I) = \mathcal{O}_M(M \setminus U)$. Hence let $g \in I$ be such that $g|_{\overline{U}^c} = f|_{\overline{U}^c}$ and let $h \in \mathcal{O}(M)$ be such that $h|_{\overline{V}^c} = 0$ and $h|_U = 1$. Then $f = f \cdot h = g \cdot h \in I$.

We are now ready to end the proof by showing that there exist integers $p_1, \dots, p_n$ such that $\mathcal{J}_{x_1}^{p_1} \cap \dots \cap \mathcal{J}_{x_n}^{p_n} \subseteq I$. Indeed notice that $I_{x^i} := \epsilon_{x^i}(I)$ is a finite codimensional ideal of $\mathcal{O}_{M,x^i}$ and hence, due to Proposition 4.3.5, contains $J_{x^i}^{p_i}$ for some integer p_i. Let now $f \in \mathcal{J}_{x_1}^{p_1} \cap \dots \cap \mathcal{J}_{x_n}^{p_n}$ and let $g_i \in I$ be such that $[f]_{x^i} = [g_i]_{x^i}$. Denote by V_i the open subsets of $|M|$ such that $f|_{V_i} = g_i|_{V_i}$, and let U_i be another family of open sets such that $U_i \subseteq \overline{U_i} \subseteq V_i$. There exists a partition of unity $\{h_1, \dots, h_n, h\}$ subordinated to the open cover $\{V_1, \dots, V_n, \overline{U_1}^c \cap \dots \cap \overline{U_n}^c\}$. Then $f = \sum f \cdot h_i + f \cdot h$ and $f \cdot h$ is in I since it is zero in a neighborhood of Z_J, while $f \cdot h_i = g_i \cdot h_i$ is obviously in I. □

The next proposition identifies the finite support distributions with the Sweedler dual.

Proposition 4.7.5. *Let $\mathcal{O}(M)'$ be the topological dual of $\mathcal{O}(M)$, then*

$$\mathcal{O}(M)^\circ = \{\phi \in \mathcal{O}(M)' \mid \operatorname{supp} \phi \textit{ consists of a finite number of points}\}.$$

Proof. ($\subset$). Let $\phi \in \mathcal{O}^\circ(M)$. Then $\ker \phi$ contains an ideal I of finite codimension and by the previous lemma we have $\ker \phi \supset I \supset \mathcal{J}_{x_1}^{p_1} \cap \dots \cap \mathcal{J}_{x_n}^{p_n}$. We now claim that $\operatorname{supp} \phi \subset \{x_1, \dots, x_n\}$. Let $x \notin \{x_1, \dots, x_n\}$ and choose open mutually disjoint neighbourhoods U, U_i of x and the x_i's, respectively. Let $f \in \mathcal{O}(M)$ with $\operatorname{supp} f \subset U$. Then $f \in \mathcal{J}_{x_i}^{p_i}$ for all i and consequently $f \in \ker \phi$, hence x is not in the support of ϕ.

($\supset$). If $\operatorname{supp} \phi = \{x_1 \dots x_n\}$, then, due to point (1) of Proposition 4.7.3, we have $\phi = \sum \phi_{x_i}$, where ϕ_{x_i} is a distribution with support at x_i. We can hence restrict ourselves to consider the case in which ϕ is a point supported distribution (i.e., its support consists of just one point). In this case the result follows easily using (2) of Proposition 4.7.3. In fact, $\ker \phi \supset \mathcal{J}_{x_i}$. □

Lemma 4.7.6. *$\mathcal{O}(M)^\circ$ separates the points of $\mathcal{O}(M)$, that is,*

$$\langle f - h, \phi_x \rangle = 0 \textit{ for all } \phi_x \in \mathcal{O}(M)^\circ \implies f = h,$$

where ϕ_x is a point supported distribution at $x \in |M|$.

Proof. Let $a = f - h$. We are going to show that if $a \neq 0$, then there exists a point supported distribution ϕ_x with $\phi_x(a) \neq 0$. Let U be an open coordinate neighborhood such that

$$a|_U = \sum_I a_I \theta^I \neq 0.$$

Clearly there exist a_J and $x \in U$ such that $a_J(x) \neq 0$. Consider the element ϕ_x of $\mathcal{O}(M)^\circ$ given by $\left(\frac{\partial}{\partial \theta^J}\right)_x$, then $\phi_x(a) \neq 0$. □

Proposition 4.7.7. *Each supermanifold morphism* $\psi : M \to N$ *uniquely determines a morphism of super coalgebras* $\psi^\circ : \mathcal{O}(M)^\circ \to \mathcal{O}(N)^\circ$ *given by* $\psi^\circ(\phi) = \phi \circ \psi^*$.

Proof. ψ° is well defined. In fact, suppose that $\phi \in \mathcal{O}(M)^\circ$. Then $\ker\phi \supseteq \mathcal{J}_{x_1}^{p_1} \cap \dots \cap \mathcal{J}_{x_n}^{p_n}$, the latter being an ideal of finite codimension. Since $|\psi^*(f)|(x^i) = f(|\psi|(x^i)) = 0$, we have $\ker\phi \circ \psi^* \supseteq \mathcal{J}_{|\psi|(x_1)}^{p_1} \cap \dots \cap \mathcal{J}_{|\psi|(x_n)}^{p_n}$. □

The above proposition thus establishes that the map

$$\mathrm{Hom}(M, N) \xrightarrow{*} \mathrm{Hom}(\mathcal{O}(M)^\circ, \mathcal{O}(N)^\circ)$$

is injective. However, in general, this map is not bijective.

4.8 Complex and real supermanifolds

In our description of the foundations of the theory of supermanifolds, we have limited our discussion to the category of differentiable supermanifolds. Most of our definitions, however, hold also in other contexts and in this section we are interested in a supergeometric theory generalizing real or complex analytic (or holomorphic) manifolds.

Definition 4.8.1. Let k be the real or the complex field and let $\mathcal{H}_{k^p}$ denote the sheaf of real or complex analytic functions on k^p. We define the *analytic superdomain* $k^{p|q}$ as the topological space k^p endowed with the following sheaf of superalgebras. For any open subset $U \subset k^p$,

$$\mathcal{O}_{k^{p|q}}(U) := \mathcal{H}_{k^p}(U) \otimes \bigwedge(\theta_1, \dots, \theta_q),$$

where $\bigwedge(\theta^1, \dots, \theta^q)$ is the exterior algebra generated by the q variables $\theta_1, \dots, \theta_q$.

Notation. In this section only, we use the symbol $\mathbb{R}^{p|q}$ to denote the real analytic superdomain instead of the C^∞ superdomain as in the rest of the text.

A *real analytic or complex analytic supermanifold* of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ which is locally isomorphic to $k^{p|q}$, i.e., for all $x \in |M|$ there exists an open neighbourhood of x $V_x \subset |M|$ and $U \subset k^p$ such that

$$\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{k^{p|q}}|_U.$$

A *morphism* of real or complex analytic supermanifolds is simply a morphism of their underlying superspaces.

Most of the results discussed in Chapters 4 and 5 hold also for real and complex analytic supermanifolds. For example, we have the chart theorem, we can give the exact same definition for tangent vector and tangent bundle to a real or a complex supermanifold and we can prove in the same way all of the theorems relative to the local structure of morphisms, like the inverse function theorem, the submersion, immersion and the constant rank morphism theorems.

There are however some important differences at the very core of the theory: partitions of unity are not available for real or complex analytic (super)manifolds, so one must exert extreme care in generalizing all the results that make use of them. Moreover, contrary to the smooth case, the superalgebra of the global sections of the structure sheaf of a (super)manifold tells us very little, in general, about the structure of the (super)manifold and in general does not allow us to retrieve all the information about the (super)manifold itself.

The main goal of this section is to understand how it is possible to define real structures and real forms in supergeometry. A major character in this game is the complex conjugate of a super manifold.

Let us first review quickly the classical setting.

Let $M = (|M|, \mathcal{H}_M)$ be a complex manifold. The *complex conjugate* of M is the manifold $\mathbb{M} = (|M|, \mathcal{H}_{\mathbb{M}})$ where $\mathcal{H}_{\mathbb{M}}$ is the sheaf of the antiholomorphic functions on M, which are immediately defined once we have $\mathcal{H}_M$ and the complex structure on M. We have the $\mathbb{C}$-antilinear sheaf morphism

$$\mathcal{H}_M \to \mathcal{H}_{\mathbb{M}}, \quad f \mapsto \bar{f}. \tag{4.5}$$

In the super context it is not possible to speak directly of antiholomorphic functions and for this reason we need the following generalization of complex conjugate super manifold.

Definition 4.8.2. Let $M = (|M|, \mathcal{O}_M)$ be a complex super manifold. We define a *complex conjugate* of M as a complex super manifold $\mathbb{M} = (|\mathbb{M}|, \mathcal{O}_{\mathbb{M}})$, where now $\mathcal{O}_{\mathbb{M}}$ is just a supersheaf, together with a super ringed space $\mathbb{C}$-antilinear isomorphism with M, which is (4.5) on the reduced supermanifold. This means that we have an isomorphism of topological spaces $|M| \cong |\mathbb{M}|$ and a $\mathbb{C}$-antilinear sheaf isomorphism

$$\mathcal{O}_M \to \mathcal{O}_{\mathbb{M}}, \quad f \mapsto \bar{f}. \tag{4.6}$$

By an abuse of notation we shall also write $f \mapsto \bar{f}$ for the inverse of the morphism (4.6).

Example 4.8.3. A complex conjugate of $M = \mathbb{C}^{1|1} = (\mathbb{C}, \mathcal{O}_{\mathbb{C}}[\theta])$ is $\mathbb{M} = (\mathbb{C}, \mathcal{O}_{\bar{\mathbb{C}}}[\bar{\theta}])$, where $\mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_{\bar{\mathbb{C}}}$ denote respectively the sheaf of holomorphic and antiholomorphic functions on $\mathbb{C}$. In fact the $\mathbb{C}$-antilinear isomorphism is

$$\mathcal{O}_M \to \mathcal{O}_{\mathbb{M}}, \quad z \mapsto \bar{z}, \quad \theta \mapsto \bar{\theta}.$$

Notice that while $\bar{z}$ has the meaning of being the complex conjugate of z, $\bar{\theta}$ is simply another odd variable that we introduce to define the complex conjugate.

Practically one can think of the complex conjugate super manifold as a way of giving a meaning to $\bar{f}$, the complex conjugate of a super holomorphic function.

Remark 4.8.4. The underlying real manifold $M_{\mathbb{R}}$ of a complex manifold M is defined via the identification of $\mathbb{C} \cong \mathbb{R}^2$ and its real dimension is double the complex dimension of M. Notice that the underlying real manifolds $M_{\mathbb{R}}$ and $\mathbb{M}_{\mathbb{R}}$ of M and $\mathbb{M}$ are isomorphic as real manifolds. If $z = (z_1, \dots, z_n)$ are complex local coordinates for M, by a slight abuse of notation we can give real local coordinates for both M and $\mathbb{M}$ as $(z, \bar{z})$ (where $\bar{z}$ is the complex conjugate).

We are ready to define a real structure on a complex supermanifold.

Definition 4.8.5. Let $M = (|M|, \mathcal{O}_M)$ be a complex super manifold and $\mathbb{M}$ a complex conjugate of M. We define a *real structure* on M as an involutive isomorphism of super ringed spaces $\rho\colon \mathbb{M} \to M$, which is $\mathbb{C}$-antilinear on the sheaves $\rho^*\colon \mathcal{O}_M \to \rho^*\mathcal{O}_{\mathbb{M}}$, $\rho^*(\lambda f) = \bar{\lambda}\rho^*(f)$. We define the *real form* M_ρ of M defined by ρ as the supermanifold $(|M|^\rho, \mathcal{O}_{M_\rho})$ where $|M|^\rho$ are the fixed points of $\rho\colon |M| \to |\mathbb{M}| = |M|$ and $\mathcal{O}_{M_\rho}$ are all the functions $f \in \mathcal{O}_M|_{|M|^\rho}$ such that $\overline{\rho^*(f)} = f$.

Remark 4.8.6. Observe that classically $M_\rho = (|M|^\rho, \mathcal{O}_{M_\rho})$ has a real manifold structure. In coordinates, the map ρ is $z \mapsto \bar{z}$. Let us look at ρ as a real differentiable map from $\mathbb{M}$ to M, seen as real manifolds. Since this is a local question, we look at ρ in a neighborhood with coordinates z and $\bar{z}$ (see Remark 4.8.4). We have

$$\rho\colon M \to \mathbb{M}, \quad (z, \bar{z}) \mapsto (\bar{z}, z).$$

We are looking at the fixed points m of ρ, $\rho(m) = m$ or $(\mathrm{id} - \rho)(m) = 0$. So $M_\rho = (\mathrm{id} - \rho)^{-1}(0)$. The differential of $(\mathrm{id} - \rho)$ is

$$\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}.$$

By the ordinary constant rank theorem we have that M_ρ is a submanifold of dimension half the real dimension of M.

We are ready to discuss examples of real structures and forms in the super context.

Example 4.8.7. We take $M = \mathbb{C}$. In this case we have $|\rho|\colon \mathbb{C} \to \mathbb{C}$, $|\rho|(z) = \bar{z}$ and $\rho^*(f(z)) = f(\bar{z})$. It is clear that $\mathbb{R} = \mathbb{C}^\rho$ and that the functions are $f \in \mathcal{O}_{\mathbb{C}}|_{\mathbb{R}}$ (where $z = \bar{z}$) such that $\bar{f} = f$.

Example 4.8.8. $M = \mathbb{C}^{1|1}$. $|\rho|$ is the same as above and $\rho^*\colon \mathcal{O}_{\mathbb{C}}[\theta] \to \mathcal{O}_{\mathbb{C}}[\bar{\theta}]$, $f(z, \theta) \mapsto f(\bar{z}, \bar{\theta})$. Again $|M|^\rho$ is the same as above while the functions are those $f \in \mathcal{O}_M|_{\mathbb{R}}$ such that $\bar{f}(\bar{z}, \bar{\theta}) = f(z, \theta)$. For example $z|_{\mathbb{R}}$ and θ are global sections satisfying such a requirement. Actually this example is particularly instructive since it shows that the odd variables come out practically unchanged from this procedure. For this reason it is immediate to see that M_ρ is a real super manifold since it is a real classical manifold and the local splitting is preserved.

If M is a complex supermanifold, one can always construct the complex conjugate $\bar{M}$ in the following way using the functor of points notation. Take $|\mathbb{M}| = |M|$ and as $\mathcal{O}_{\mathbb{M}}$ the sheaf $\mathcal{O}_M$ with the complex conjugate $\mathbb{C}$-algebra structure (that is $\lambda \cdot f = \bar{\lambda} f$). In order to obtain a real structure on M, we need a ringed spaces morphism $M \to \mathbb{M}$ with certain properties. By Yoneda's lemma this is equivalent to give an invertible natural transformation between the functors of points:

$$\rho\colon M(R) \to \mathbb{M}(R)$$

for all super ringed spaces R satisfying the $\mathbb{C}$-antilinear condition.

4.9 References

The idea of supercalculus goes back to Berezin's seminal work [10]. The concept of supermanifold was however introduced later by Berezin and Leites in [11] and in the beautiful work [49] by Kostant. In particular [49] can be considered as the first systematic and rigorous treatment of the foundations of smooth supergeometry. The approach there is very algebraic in nature with a particular attention to the coalgebra structure of the structure sheaf of a supermanifold. Section 4.7 provides a bridge between Kostant's approach and ours.

Our treatment of superdomains, in particular Lemma 4.1.9 and the Chart Theorem 4.1.11, follows quite closely [53]. The content of Section 4.5 and our treatment of the localization procedure in the super-setting is found also in [6]. In Section 4.7, we generalize the proof of Proposition 4.7.5 given in [58].

For further reading and a more modern treatment of smooth supergeometry the reader can consult [56], [6], [22], [76].

5

The local structure of morphisms

The goal of this chapter is to study the local structure of supermanifold morphisms. As in the ordinary setting we have the inverse function theorem and locally we can classify the morphisms according to the rank of their differentials, so that we can formulate the supergeometric versions of the immersion and submersion theorems. There is however an important difference with the ordinary differentiable manifolds theory, which arises when we discuss the *constant rank morphisms*, which are the natural generalization of the immersion and submersion morphisms. There is in fact a difficulty with the definition of rank of the Jacobian of a morphism: as we shall see this notion is not always well defined. This is a supergeometric peculiarity, ultimately linked to the graded nature of super vector spaces. The constant rank morphisms are crucial for a complete treatment of submanifolds and it is the key tool for their explicit constructions as we shall see in the section on submanifolds.

All of these are well-known results that appeared in several papers, including [53], [56], [76]. Nevertheless we provide complete proofs of them in the effort to make the text self-contained.

5.1 The inverse function theorem

We start by proving the super analogue of the inverse function theorem. As in the ordinary case it is of fundamental importance. Its proof heavily relies on the classical version of the theorem, for which we refer to [52].

Proposition 5.1.1 (The inverse function theorem). *Let $\phi\colon M \to N$ be a supermanifold morphism and let $m \in |M|$ such that $(\mathrm{d}\phi)_m$ is bijective. Then there exist charts U and V around m and $|\phi|(m)$ respectively such that $|\phi|(U) \subseteq V$ and $\phi|_U\colon U \to V$ is an isomorphism of U onto V.*

Proof. Since the statement is local we can assume both M and N to be superdomains $U^{p|q}$ and $V^{p|q}$ respectively. The superdimensions must be equal since the differential is bijective.

Denote by x^i, ξ^j and t^r, θ^s supercoordinates on $U^{p|q}$ and $V^{p|q}$, respectively. By the Chart Theorem 4.1.11, $\phi^*\colon \mathcal{O}(V^{p|q}) \to \mathcal{O}(U^{p|q})$ is given by

$$\begin{cases} \phi^*(t^r) = \sum_{|P|\geq 0} \phi^r_P(x^1,\dots,x^p)\xi^P, \\ \phi^*(\theta^s) = \sum_{|Q|\geq 1} \Phi^s_Q(x^1,\dots,x^p)\xi^Q. \end{cases}$$

Our assumption on the differential being bijective amounts to saying that

$$A := \begin{pmatrix} \frac{\partial \phi_0^1}{\partial x^1} & \cdots & \frac{\partial \phi_0^1}{\partial x^p} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_0^p}{\partial x^1} & \cdots & \frac{\partial \phi_0^p}{\partial x^p} \end{pmatrix}, \quad B := \begin{pmatrix} \Phi^1_{(1,0,\dots,0)} & \cdots & \Phi^1_{(0,0,\dots,1)} \\ \vdots & \ddots & \vdots \\ \Phi^p_{(1,0,\dots,0)} & \cdots & \Phi^p_{(0,0,\dots,1)} \end{pmatrix}$$

are non-singular. We can hence apply the classical inverse function theorem to find an open subset $W^{p|q} \subseteq U^{p|q}$ where $\{\phi_0^i, \sum_k B_{jk}\xi^j\}$ define a new supercoordinate system that we denote by $\{\bar{x}^i, \bar{\xi}^j\}$. If we denote by τ the corresponding change of coordinates we have

$$\begin{cases} (\tau^* \circ \phi^*)(t^i) = \bar{x}^i + \sum_{|P|\geq 2} \bar{\phi}^i_P(\bar{x}^1, \dots, \bar{x}^p)\bar{\xi}^P, \\ (\tau^* \circ \phi^*)(\theta^j) = \bar{\xi}^j + \sum_{|Q|\geq 3} \bar{\Phi}^j_Q(\bar{x}^1, \dots, \bar{x}^p)\bar{\xi}^Q. \end{cases}$$

Hence $\tau^* \circ \phi^*$ has the form $\mathrm{id} + N$, where N is a matrix with nilpotent entries. This implies at once that $\tau^* \circ \phi^*$ has an inverse ψ^* given by $\sum (-1)^j N^j$ where the sum terminates due exactly to the nilpotency of N. Hence ϕ^* is invertible. Consequently, by the Chart Theorem, ϕ is an isomorphism. □

Definition 5.1.2. Let $\phi\colon M \to N$ be a supermanifold morphism. If $(\mathrm{d}\phi)_m$ is bijective, we say that ϕ is a *local superdiffeomorphism* at $m \in |M|$. If $(\mathrm{d}\phi)_m$ is bijective for all $m \in |M|$, we say that ϕ is a local superdiffeomorphism.

If ϕ is invertible and its inverse is a supermanifold morphism, we say that ϕ is a *superdiffeomorphism*. To simplify matters we shall often drop the suffix "super" and speak of "diffeomorphism" whenever it is clear in which category we are working.

The next proposition clarifies the relation between local superdiffeomorphism and superdiffeomorphism.

Corollary 5.1.3. *Suppose that $\phi\colon M \to N$ is a supermanifold morphism such that*

(1) $|\phi|$ *is bijective,*

(2) $(\mathrm{d}\phi)_x$ *is an isomorphism for each* $x \in |M|$.

Then ϕ is a superdiffeomorphism.

Proof (Sketch). We shall produce a supermanifold morphism $\psi = (|\psi|, \psi^*)\colon N \to M$, which is the inverse of ϕ. We have by $|\psi| = |\phi^{-1}|$ by (1). As for the sheaf morphism, ϕ^* is locally invertible, so we can define $\psi^*_U = {\phi^*}^{-1}|_U$ on any U in a suitable open cover of N. Such morphisms agree on the overlap of open sets, thus we have defined the required inverse. □

5.2 Immersions, submersions and the constant rank morphisms

When M and N are classical manifolds and $\phi\colon M \to N$ is a supermanifold morphism, there exists a relationship between the properties of the differential $(\mathrm{d}\phi)_x$ at a point $x \in |M|$ and the local structure of the map ϕ in a neighborhood of x. This leads to a classification of smooth mappings in terms of the rank of their differentials.

The purpose of this section is to replicate this same discussion in the supergeometric setting. As we already pointed out, there are important differences with the ordinary case in the definition of constant rank morphism.

Let us recall that given a morphism $\phi\colon M \to N$, we can speak both of its differential $(\mathrm{d}\phi)_m$ at $m \in |M|$ and of the Jacobian J_ϕ in a neighborhood of $m \in |M|$. Choosing charts (U, t, θ) and (V, x, ξ) containing m and $|\psi|(m)$, respectively, the differential and the Jacobian can be written as

$$(\mathrm{d}\phi)_m = \begin{pmatrix} \frac{\partial\phi^*(x)}{\partial t}\big|_m & 0 \\ 0 & \frac{\partial\phi^*(\xi)}{\partial\theta}\big|_m \end{pmatrix}, \quad J_\phi = \begin{pmatrix} \frac{\partial\phi^*(x)}{\partial t} & -\frac{\partial\phi^*(x)}{\partial\theta} \\ \frac{\partial\phi^*(\xi)}{\partial t} & \frac{\partial\phi^*(\xi)}{\partial\theta} \end{pmatrix}.$$

Clearly we have the relation $\widetilde{J_\phi}(m) = (\mathrm{d}\phi)_m$.

In Chapter 1, Section 1.5, we have defined the rank of a supermatrix as the dimension of the largest invertible supermatrix contained in it. With this definition one has immediately that the rank of a supermatrix coincides with the rank of its reduced matrix, so in particular $\mathrm{rk}(J_\phi) = \mathrm{rk}(\mathrm{d}\phi)_m$.

We are now going to introduce a definition that is a variation of the concept of rank and is subtly related to it.

Definition 5.2.1. Let $Z = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right) \in \mathrm{M}_{p|q\times m|n}(A)$ for a commutative superalgebra A; in other words, Z is a $p|q \times m|n$ matrix with diagonal blocks entries in A_0 and off diagonal block entries in A_1 (our prototype for A is $\mathcal{O}_M(U)$). We say that Z has *constant rank* $r|s$ if there exist $G_1 \in \mathrm{GL}_{m|n}(A)$ and $G_2 \in \mathrm{GL}_{p|q}(A)$ such that $G_1 Z G_2$ has the form

$$G_1 Z G_2 = \begin{pmatrix} \mathrm{id}_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 5.2.2. It is important to notice that the constant rank is not defined for all supermatrices. For example, for the supermatrix $Z = \left(\begin{smallmatrix} 0 & 0 \\ \xi & 0 \end{smallmatrix}\right) \in \mathrm{M}_{1|1\times 1|1}$ we cannot define any constant rank; in fact, as one can readily check, it is not possible with a basis change, that is, with left and right multiplication by elements of the general linear supergroup, to transform Z into a diagonal matrix with 1 or 0 on the diagonal, as the definition of constant rank requires. However, once the constant rank is defined, it coincides with the rank as we defined it in Chapter 1, Section 1.5. We leave this check to the reader as an exercise.

We can now give the following definitions.

Definition 5.2.3. Let the notation be as above.

(1) A morphism $\phi\colon M \to N$ is said to be an *immersion* at $m \in |M|$ if $(\mathrm{d}\phi)_m$ is injective.

(2) A morphism $\phi\colon M \to N$ is said to be a *submersion* at $m \in |M|$ if $(\mathrm{d}\phi)_m$ is surjective.

(3) A morphism $\phi\colon M \to N$ is said to have constant rank $r|s$ in a neighborhood U of $m \in |M|$, if J_ϕ is a matrix of *constant rank* $r|s$ (and entries in $\mathcal{O}_M(U)$).

Notice that in the first two definitions we refer to the differential of the morphism, while in the last one the Jacobian of the morphism enters.

Each one of the three kinds of morphisms has its corresponding structure theorem and, as we shall see, there is an important result relating all three of them. We now begin our discussion of each of these three kinds of morphisms.

Immersions. Let $\phi\colon M \to N$ be a morphism of supermanifolds, $\dim M = m|n$, $\dim N = m + p|n + q$ and let $x \in |M|$. Suppose that there exist charts U, $\{t^i\}_{i=1}^m$, $\{\theta^j\}_{j=1}^n$ around x, and $V = V_1 \times V_2$, $\{y^i\}_{i=1}^m$, $\{z^a\}_{a=1}^p$, $\{\eta^j\}_{j=1}^n$, $\{\zeta^b\}_{b=1}^q$ around $|\phi|(x)$ such that the restriction of ϕ to U has the form

$$y^i \mapsto t^i, \quad \eta^j \mapsto \theta^j, \quad z^a, \zeta^b \mapsto 0.$$

By a common abuse of notation, whenever there is no danger of confusion, we shall denote from now on the coordinates in the domain and their images by the same letter, so we shall write y^i and η^j instead of t^i and θ^j. Clearly ϕ is an immersion at x. The converse is also true and this is the content of the following proposition.

Proposition 5.2.4. *Let $\phi\colon M \to N$ be a supermanifold morphism, with* $\dim(M) = m|n \leq \dim(N) = m + p|n + q$. *The following facts are equivalent:*

(1) *$\phi\colon M \to N$ is an immersion at x.*

(2) *$(\mathrm{d}\phi)_x$ has rank $(m|n)$.*

(3) *There exist charts U, $\{t^i\}_{i=1}^m$, $\{\theta^j\}_{j=1}^n$ around x and $V = V_1 \times V_2$, $\{t^i\}_{i=1}^m$, $\{\bar{t}^a\}_{a=1}^p$, $\{\theta^j\}_{j=1}^n$, $\{\bar{\theta}^b\}_{b=1}^q$ around $|\phi|(x)$ such that the restriction of the map to U and V has the form*

$$t^i \mapsto t^i, \quad \theta^j \mapsto \theta^j, \quad \bar{t}^a, \bar{\theta}^b \mapsto 0.$$

Proof. The proof mimics the classical one. We sketch it briefly. The equivalence between (1) and (2) is true by definition, while (3) implies (1) comes from our previous discussion.

Let $(W, \{y^i\}, \{\eta^j\})$ and $(Q, \{x^r\}, \{\xi^s\})$ be supercharts near x and $|\phi|(x)$, respectively. Possibly relabeling the supercoordinates we can suppose that $\left(\frac{\partial\phi^*(x^r)}{\partial y^i}\right)_{r,i=1}^m$ and $\left(\frac{\partial\phi^*(\xi^s)}{\partial\eta^j}\right)_{s,j=1}^n$ are non-singular at x.

Hence if we consider the superchart $(\mathbb{R}^{p|q}, z^a, \zeta^b)$, we can define the map

$$\alpha\colon W \times \mathbb{R}^{p|q} \to Q$$

through $\alpha^*(x^r) = \phi^*(x^r)$ for $1 \leq r \leq m$, $\alpha^*(x^r) = z^{r-m} + \phi^*(x^r)$ for $m+1 \leq r \leq m+p$, $\alpha^*(\xi^s) = \phi^*(\xi^s)$ for $1 \leq s \leq n$, and $\alpha^*(\xi^s) = \zeta^{s-n} + \phi^*(\xi^s)$ for $n+1 \leq s \leq n+q$. It is clear that $\phi = \alpha \circ i_{W \hookrightarrow W \times \mathbb{R}^{p|q}}$. It is also clear that $(\mathrm{d}\alpha)_{(x,0)}$ is a bijection so that, due to Proposition 5.1.1, it is a local diffeomorphism. Thus the proposition is proved. □

Submersions. Let us proceed in analogy with immersions.

Proposition 5.2.5. *Let $\phi\colon M \to N$ be a supermanifold morphism, with* $\dim(M) = m+p|n+q \geq \dim(N) = m|n$. *The following facts are equivalent:*

(1) *$\phi\colon M \to N$ is a submersion at x of rank $m|n$.*

(2) *$(\mathrm{d}\phi)_x$ has rank $(m|n)$.*

(3) *There exist charts $U = U_1 \times U_2$, $\{t^i\}_{i=1}^m$, $\{\bar{t}^a\}_{a=1}^p$, $\{\theta^j\}_{j=1}^n$, $\{\bar{\theta}^b\}_{b=1}^q$ around x, and V, $\{t^i\}_{i=1}^m$, $\{\theta^j\}_{j=1}^n$ around $|\phi|(x)$ such that the restriction of the map to U and V has the form*

$$t^i \mapsto t^i, \quad \theta^j \mapsto \theta^j.$$

Proof. The proof is completely similar to that for immersion and is left out. □

Constant rank morphisms. Due to the fact that the definition of the constant rank morphisms involves the Jacobian matrix rather than the differential, the discussion is more involved.

Proposition 5.2.6. *Let M and N be supermanifolds,* $\dim M = p|q$, $\dim N = m|n$. *Suppose that $\phi\colon M \to N$ is a constant rank morphism of rank $\alpha|\beta$ in a neighborhood of $x \in |M|$. Then there exist charts $U \simeq U^{\alpha|\beta} \times U^{p-\alpha|q-\beta}$ with coordinates $(y^i, z^j, \eta^r, \zeta^s)$ and $V \simeq V^{\alpha|\beta} \times V^{m-\alpha|n-\beta}$ with coordinates $(y^i, t^k, \eta^r, \theta^l)$ containing x and $|\phi|(x)$, respectively, such that ϕ^* has the form*

$$y^i \mapsto y^i, \quad \eta^r \mapsto \eta^r,$$
$$t^k \mapsto 0, \quad \theta^l \mapsto 0.$$

Proof. Since the statement is local, we can work on superdomains. Suppose that we have a morphism $\phi\colon U^{p|q} \to V^{m|n}$ and let (u, μ), (v, ν) be local coordinates in $U^{p|q}$ and $V^{m|n}$. By the Chart Theorem we have that ϕ is described by the pullbacks $v^{i*} := \phi^*(v^i)$ and $\nu^{j*} := \phi^*(\nu^j)$. It is not restrictive to suppose that 0 belongs both to $U^{p|q}$ and to $V^{m|n}$ and that $v^{i*}(0) = \nu^{j*}(0) = 0$. Since J_ϕ has constant rank, possibly relabeling the coordinates, we can assume that the matrices

$$\left(\frac{\partial v^{i*}}{\partial u^r}\right)_{i,r=1}^{\alpha} \quad \text{and} \quad \left(\frac{\partial \nu^{j*}}{\partial \mu^s}\right)_{j,s=1}^{\beta}$$

are non-singular. In order to keep the notation minimal let w and ρ generically denote the supercoordinates in a superdomain. Hence the morphism

$$\psi\colon U^{p|q} \to \mathbb{R}^{\alpha|\beta} \times \mathbb{R}^{p-\alpha|q-\beta},$$

defined by the pullbacks $\psi^*(w^i \otimes 1) = v^{i*}$ for $1 \le i \le \alpha$, $\psi^*(1 \otimes w^j) = u^{\alpha+j}$ for $1 \le j \le p-\alpha$, $\psi^*(\rho^r \otimes 1) = \nu^{r*}$ for $1 \le r \le \beta$, and $\psi^*(1 \otimes \rho^s) = \mu^{\beta+s}$ for $1 \le s \le q-\beta$, is a local superdiffeomorphism. An easy check shows indeed that ψ has invertible differential at x. Hence there exist $U_1 \subseteq U^{p|q}$ and $U_2 \subseteq \mathbb{R}^{\alpha|\beta} \times \mathbb{R}^{p-\alpha|q-\beta}$ such that $\psi\colon U_1 \to U_2$ is an isomorphism. The morphism $\phi \circ \psi^{-1}$ is determined by the pullback

$$\begin{aligned}
&v^i \mapsto w^i \otimes 1, && 1 \le i \le \alpha,\\
&v^i \mapsto g^j(w^1 \otimes 1, 1 \otimes w^j, \rho^r \otimes 1, 1 \otimes \rho^s), && \alpha+1 \le j \le m,\\
&\nu^r \mapsto \rho^r \otimes 1, && 1 \le r \le \beta,\\
&\nu^s \mapsto \gamma^s(w^1 \otimes 1, 1 \otimes w^j, \rho^r \otimes 1, 1 \otimes \rho^s)1, && \beta+1 \le s \le n.
\end{aligned}$$

The Jacobian matrix is hence given by

$$J_{\phi\circ\psi^{-1}} = \begin{pmatrix} I & 0 & 0 & 0 \\ \frac{\partial g}{\partial w_1} & \frac{\partial g}{\partial w_2} & \frac{\partial g}{\partial \rho_1} & \frac{\partial g}{\partial \rho_2} \\ 0 & 0 & I & 0 \\ \frac{\partial \gamma}{\partial w_1} & \frac{\partial \gamma}{\partial w_2} & \frac{\partial \gamma}{\partial \rho_1} & \frac{\partial \gamma}{\partial \rho_2} \end{pmatrix},$$

where we have denoted collectively by w_1 (resp. w_2, ρ_1, ρ_2) the variables of the form $w^i \otimes 1$ (resp. $1 \otimes w^i$, $\rho^i \otimes 1$, $1 \otimes \rho^i$).

It is clearly possible to rearrange rows and columns so that the matrix takes the more tractable form

$$J_{\phi\circ\psi^{-1}} := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \frac{\partial g}{\partial w_1} & \frac{\partial g}{\partial \rho_1} & \frac{\partial g}{\partial w_2} & \frac{\partial g}{\partial \rho_2} \\ \frac{\partial \gamma}{\partial w_1} & \frac{\partial \gamma}{\partial \rho_1} & \frac{\partial \gamma}{\partial w_2} & \frac{\partial \gamma}{\partial \rho_2} \end{pmatrix}.$$

Notice that correspondingly also the usual form of the matrices in the general linear supergroup will change. In other words, when we write matrices in $\mathrm{GL}(V(R))$, where we choose for V a homogeneous basis where the even elements do not all come at the beginning, then we no longer have matrices with even entries in the diagonal blocks and odd entries in the off diagonal blocks. So these matrices represent elements in $\mathrm{GL}(V(R))$, but they are not in $\mathrm{GL}_{k|l}(R)$, for a given commutative superalgebra R.

The claim is now that, in order for the matrix $J_{\phi\circ\psi^{-1}}$ to have constant rank $\alpha|\beta$, it is necessary and sufficient that the submatrix $\begin{pmatrix} \frac{\partial g}{\partial w_2} & \frac{\partial g}{\partial \rho_2} \\ \frac{\partial \gamma}{\partial w_2} & \frac{\partial \gamma}{\partial \rho_2} \end{pmatrix}$ is zero. Suppose indeed that

$M := \begin{pmatrix} I_{\alpha,\beta} & 0 \\ A & B \end{pmatrix}$ is a matrix of constant rank $\alpha|\beta$, where $I_{\alpha,\beta}$ is the identity matrix with $\alpha+\beta$ rows and columns. Hence there exists G and $\overline{G}$ such that $\overline{G}MG = \begin{pmatrix} I_{\alpha,\beta} & 0 \\ 0 & 0 \end{pmatrix}$. In particular there exists $G = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$ such that MG has only the first $\alpha+\beta$ columns non-zero. In our case we have

$$MG = \begin{pmatrix} G_1 & G_2 \\ AG_1 + BG_3 & AG_2 + BG_4 \end{pmatrix}.$$

So if $G_2 = 0$, then G_4 is invertible and consequently $B = 0$.

We thus have that g^i and γ^j do not depend on the coordinates w_2, ρ_2 on $\mathbb{R}^{p-\alpha|q-\beta}$. We can hence simplify the notation and write

$$g^i(w^i \otimes 1, \rho^r \otimes 1), \quad \alpha+1 \le i \le m,$$
$$\gamma^j(w^i \otimes 1, \rho^r \otimes 1), \quad \beta+1 \le j \le n.$$

Consider now the superdomain $U_3 \subseteq V^{m|n}$ given by

$$U_3 := \{(v^1,\dots,v^m,\nu^1,\dots,\nu^n) \mid (v^1,\dots,v^\alpha,0,\dots,0,\nu^1,\dots,\nu^\beta,0,\dots,0) \in U_2\}$$

and define the morphism

$$\varphi\colon U_3 \to \mathbb{R}^{\alpha|\beta} \times \mathbb{R}^{m-\alpha|n-\beta}$$

via

$$w^i \otimes 1 \mapsto v^i, \qquad \rho^r \otimes 1 \mapsto \nu^r,$$
$$1 \otimes w^j \mapsto v^{\alpha+j} - g^{(\alpha+j)}(v^i,\nu^j), \quad 1 \otimes \rho^s \mapsto \nu^{\beta+s} - \gamma^{(\beta+s)}(v^i,\nu^j).$$

It is a simple check to show that φ is invertible and that $\varphi \circ \phi \circ \psi^{-1}$ has the required form. □

We have immediately the following interesting corollaries.

Corollary 5.2.7. *Let $\phi\colon M \to N$ be a morphism of supermanifolds and let $m \in |M|$. Then the following are equivalent:*

(1) *In a neighbourhood of m, ϕ splits as a submersion ϕ_1 and an immersion ϕ_2*

$$\phi|_U\colon U \xrightarrow{\phi_1} V \xrightarrow{\phi_2} W.$$

(2) *ϕ is a constant rank morphism in a neighbourhood of m.*

Corollary 5.2.8. *Immersions and submersions are constant rank morphisms.*

5.3 Submanifolds

In this section we shall use the results on the local structure of morphisms discussed in the previous section in order to formulate a theory of smooth submanifolds of supermanifolds.[1] As we stressed from the very beginning, even though many classical results and constructions carry over to the super setting, the arguments present some extra subtleties and we invite the reader to go to [54] for some insight on all the problems that can arise.

As in the classical theory, submanifolds of a given supermanifold M are defined as pairs (N, j) where N is a supermanifold and $j : N \to M$ is an injective morphism with some regularity property. We will distinguish two kinds of submanifolds according to the properties of the morphism j.

Definition 5.3.1. We say that (N, j) is an *immersed submanifold* if $j : N \to M$ is an injective immersion, in other words, if $|j|: |N| \to |M|$ is injective and $(\mathrm{d}j)_m$ is injective for all $m \in |M|$.

As in the ordinary setting, we can strengthen this notion by introducing the notion of *embedding*.

Definition 5.3.2. We say that $j : N \to M$ is an *embedding* if it is an immersion and if $|j|: |N| \to |M|$ is a homeomorphism onto its image. We say that (N, j) is an *embedded submanifold* if j is an embedding. We say that (N, j) is a *closed embedded submanifold* if it is an embedded submanifold and $|j|(|N|)$ is a closed subset of $|M|$.

Remark 5.3.3. Notice that the morphism $j : \widetilde{M} \to M$, where $\widetilde{M}$ is the reduced manifold associated with M, is a closed embedding.

Closed embedded submanifolds have the following nice characterization in terms of the properties of the pullback.

Proposition 5.3.4. *Let N and M be supermanifolds. A map $j : N \to M$ is a closed embedding if and only if $j^* : \mathcal{O}(M) \to \mathcal{O}(N)$ is surjective.*

Proof. Suppose first that j is a closed embedding and let $f \in \mathcal{O}(N)$. Denote by $\{U_\alpha\}_{\alpha\in A}$ an open covering of $|N|$. Since $|j|$ is a homeomorphism onto its image, there exist open sets $\{U'_\alpha\}_{\alpha\in A}$ in $|M|$ such that $|j|(U_\alpha) = U'_\alpha \cap |j|(|N|)$. Possibly passing to a refinement of the open cover, it is not restrictive to suppose each U'_α to be a superchart with coordinates x^i, y^j, θ^p, η^q such that $j^*(y^j) = j^*(\eta^q) = 0$ (see Proposition 5.2.4). In particular the map $j^*_{U'_\alpha} : \mathcal{O}_M(U'_\alpha) \to \mathcal{O}_N(U_\alpha)$ is surjective. Let g_α be a section in $\mathcal{O}_M(U'_\alpha)$ such that $j^*_{U'_\alpha}(g_\alpha) = f_\alpha := f|_{U_\alpha}$. Since $j(|N|)$ is a closed subset of $|M|$, we can consider the open cover of $|M|$ given by $\{U'_\alpha, \alpha \in A, |M| \setminus |j|(|N|)\}$ and let $\{h_\alpha\}$ denote a partition of unity subordinated to the given

[1] To ease the terminology we prefer to use the term "submanifold" instead of the more appropriate, but cumbersome "subsupermanifold".

cover. Clearly $g_\alpha \cdot h_\alpha$ belongs to $\mathcal{O}(M)$ and, due to the local finiteness of the cover, $g := \sum_\alpha g_\alpha \cdot h_\alpha$ is a well-defined section in $\mathcal{O}(M)$ so that $j^*(g) = \sum j^*(h_\alpha) f_\alpha$. On the other hand it is very easy to check that the family $\{j^*(h_\alpha)\}$ defines a partition of unity on N subordinated to $\{U_\alpha\}$, hence $f = \sum j^*(h_\alpha) f_\alpha$. So we have proved the surjectivity of j^*.

Suppose now that the pullback $j^*\colon \mathcal{O}(M) \to \mathcal{O}(N)$ is surjective. It is immediate that both (see notation of Chapter 4, Section 4.5)

$$|j|\colon |N| \to |M|, \quad (\mathrm{d}j)_x\colon T_x N \to T_{|j|(x)} M,$$

are injective. The fact that $|j|$ is a homeomorphism onto its image is a consequence of the same result being true for the ordinary setting (see [78], Ch. I). □

Proposition 5.3.4 characterizes closed embeddings among all possible supermanifold morphisms $\phi\colon N \to M$ in terms of surjectivity of ϕ^*. As a consequence we have $\mathcal{O}(N) \simeq \mathcal{O}(M)/\mathcal{J}_N$ where $\mathcal{J}_N := \ker \phi^*$. Hence the ideal $\mathcal{J}_N \subseteq \mathcal{O}(M)$ completely characterizes the closed embedded submanifold N. The next proposition singles out some important properties of $\mathcal{J}_N$.

Proposition 5.3.5. *Let $j: N \to M$ be a closed embedding and let $J_N = \ker(j^*)$. Then if $m \in |M|$ and J_m is the corresponding maximal ideal in $\mathcal{O}(M)$, $J_N \subseteq J_m$ if and only if $m \in |j|(|N|)$. Moreover:*

(1) *If $m \in |M|$ is such that $J_N \subseteq J_m$, then there exist homogeneous $f_1, \dots, f_n$ in J_N such that $[f_1], \dots, [f_n]$ generate $J_{N,m}$ and $(\mathrm{d}f_1)_m, \dots, (\mathrm{d}f_n)_m$ are linearly independent at m, where $J_{N,m}$ denotes the ideal generated by the image of J_N in the stalk $\mathcal{O}_{M,m}$.*

(2) *If $\{f_i\}_{i\in\mathbb{N}}$ is a family in J_N such that any compact subset of M intersects only a finite number of* supp f_i, *then $\sum_i f_i$ belongs to J_N.*

Proof. We give a sketch leaving the details to the reader. Since j is a closed embedding, due to Proposition 5.3.4, $\mathcal{O}(N) \simeq \mathcal{O}(M)/J_N$ so that there is a bijective correspondence between maximal ideals in $\mathcal{O}(N)$ and maximal ideals in $\mathcal{O}(M)$ containing J_N. We now consider (1). Let $x \in N$. Since j is an immersion, there exist supercharts $(U, x^i, \theta^j) \ni x$ and $(U \times W, x^i, y^r, \theta^j, \eta^s) \ni |j|(x)$ such that j is expressed as an injection, where the coordinates y^r and η^s are sent to zero. Let $U' \times W' \ni x$ be an open subset of $U \times W$ and consider a section g such that $g|_{U' \times W'} = 1$ and $g|_{\overline{U \times W}^c} = 0$. The global section g allows us to extend the local coordinates to obtain global sections; in fact the global sections $\{y^r g, \eta^s g\}$ have the required properties to be the f_i's.

We now go about the proof of (2). We leave it to the reader to check that if $\{f_i\}$ are sections in $\mathcal{O}(M)$ such that $\{\text{supp } f_i\}$ is a locally finite covering, then the possibly infinite sum $f = \sum_i f_i$ is a well-defined element in $\mathcal{O}(M)$. Now notice that $J_N(U) := \ker j^*_{|U|}$ is an ideal sheaf, hence to check if $f \in J_N = J_N(|N|)$ it is enough to check that $f|_{U_i} \in J_N(U_i)$ for an open cover $\{U_i\}$ of $|N|$. Since any point has a compact neighbourhood and since the hypothesis in (2) holds, we obtain the result. □

Definition 5.3.6. Let I be an ideal in $\mathcal{O}(M)$. I is called a *regular ideal* if the following holds.

(1) If $m \in |M|$ is such that $I \subseteq J_m$, then there exist homogeneous $f_1, \dots, f_n$ in I such that $[f_1], \dots, [f_n]$ generate I_m and $(\mathrm{d} f_1)_m, \dots, (\mathrm{d} f_n)_m$ are linearly independent at m, where I_m denotes the ideal generated by the image of I in the stalk $\mathcal{O}_{M,m}$.

(2) If $\{f_i\}_{i \in \mathbb{N}}$ is a family in I such that any compact subset of M intersects only a finite number of supp f_i, then $\sum_i f_i$ belongs to I.

Remark 5.3.7. If the sum $\{f_i\}_{i \in \mathbb{N}}$ is finite, the second condition is trivial since it is clear that a finite sum of elements in I still belongs to the ideal I. When the sum is infinite, the sum is still well defined since at any point we are summing a finite number of f_i's. However the f_i's that we are summing can vary from point to point and consequently in general we cannot assume that their sum still lies in I.

The next proposition says that each regular ideal I in $\mathcal{O}(M)$ is of the form J_N for a closed embedded subsupermanifold N. This is a converse to Proposition 5.3.5.

Proposition 5.3.8. *Let M be a supermanifold and suppose that J is a regular ideal in $\mathcal{O}(M)$. Then there exists a unique closed embedded supermanifold (N, j) such that J is the associated regular ideal, i.e., $\mathcal{O}(N) = \mathcal{O}(M)/J$.*

Proof. Define the set

$$|N| := \{x \in |M| \mid J \subseteq J_x\}.$$

Clearly $|N| = \bigcap \{|f|^{-1}(0) \mid f \in J\}$ so that $|N|$ is closed in $|M|$. Since J is a regular ideal, for each $x \in |N|$ there exist homogeneous sections $f^1, \dots, f^p, \gamma^1, \dots, \gamma^q$ in J, depending on x, of even and odd parity, respectively, such that $\{(\mathrm{d} f^i)_x, (\mathrm{d}\gamma^j)_x\}$ are linearly independent and $[f^i], [\gamma^j]$ generate the space of germs J_x. Because of the inverse function theorem, there exist sections y^r, η^s in $\mathcal{O}(M)$ such that $\{f^i, y^r, \gamma^j, \eta^s\}$ is a coordinate system in a neighborhood U^x of x in $|M|$.

Define now $U'^x := U^x \cap |N|$ and

$$\mathcal{O}(U'^x) := C^\infty(y^1, \dots, y^n) \otimes \Lambda(\eta^1, \dots, \eta^m).$$

Then U'^x is a subset of $|M|$ of the form

$$U'^x := \{z \in |M| \mid f^i(z) = 0\}.$$

We now show that $|N|$ can be endowed with a supermanifold structure. Clearly we have the submersion

$$\alpha_x \colon \mathcal{O}(U'^x) \hookrightarrow \mathcal{O}_M(U^x)$$

given by $y^r \mapsto y^r$ and $\eta^s \mapsto \eta^s$, and there is also the immersion

$$\beta_x \colon \mathcal{O}_M(U^x) \to \mathcal{O}(U'^x), \quad f^i, \gamma^j \mapsto 0, \quad y^r \mapsto y^r, \quad \eta^s \mapsto \eta^s.$$

We can thus define transition functions $\bar{\phi}_{xy}$ between the various $\mathcal{O}(U^{x\prime})$ through the diagram

$$\begin{array}{ccc} \mathcal{O}(U'^x) & \overset{\bar{\phi}_{xy}}{\dashrightarrow} & \mathcal{O}(U'^y) \\ \alpha_x \downarrow & & \uparrow \beta_y \\ \mathcal{O}_M(U^x) & \xrightarrow{\phi_{xy}} & \mathcal{O}_M(U^y). \end{array}$$

It is evident that $\bar{\phi}_{xx} = \mathrm{id}$ and it is easy to prove that they obey the other glueing conditions as specified in Chapter 2, Section 2.2. We call N the supermanifold obtained in this way. It is clear that it is a closed embedded submanifold. □

We now introduce the concept of *quasi-coherent sheaves* on supermanifolds. As we shall see, this concept is strictly related to closed embedded submanifolds, and this terminology is widely used in the literature. We warn the reader that the word "quasi-coherent" is used to stress the analogy with quasi-coherent sheaves of ideals in the algebraic setting, but that this analogy has differences, which should not be overlooked.

Definition 5.3.9. Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold and let $\mathcal{J}$ an ideal sheaf. We say that $\mathcal{J}$ is a *quasi-coherent sheaf of ideals* if for each point $m \in |M|$, there exists an open neighbourhood U such that $\mathcal{J}(U)$ is a regular ideal.

Proposition 5.3.10. *Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold. There is a one-to-one correspondence between the following sets:*

(1) *The set of closed embedded submanifolds of M.*

(2) *The set of regular ideals of $\mathcal{O}(M)$.*

(3) *The set of quasi-coherent sheaves on M.*

Proof. The correspondence between (1) and (2) is given by Propositions 5.3.8 and 5.3.5. As for (2) and (3), it is clear that if $\mathcal{J}$ is a quasi-coherent ideal sheaf, then its global sections define a regular ideal. Vice versa if I is a regular ideal in $\mathcal{O}(M)$, then for each point $m \in |M|$ there exists an open set U containing x and homogeneous sections (y^i, η^s) in $\mathcal{O}_M(U)$ that generate a regular ideal $I(U)$ in $\mathcal{O}_M(U)$. These sections are those computed during the proof of Proposition 5.3.5. The assignment $U \mapsto I(U)$ is a sheaf of ideals on $|M|$. □

The next concept we want to introduce is *transversality*. The main way to define a submanifold of a given manifold is through equations. We want to give a criterion to establish when we can give a supermanifold structure to the set of points which are the zeros of a certain set of equations.

Definition 5.3.11. Let N_1, N_2 and M be supermanifolds. Suppose that morphisms $\phi_1 \colon N_1 \to M$ and $\phi_2 \colon N_2 \to M$ are given such that, if $m = |\phi_1|(n_1) = |\phi_2|(n_2)$, then

$$T_m M = (\mathrm{d}\phi_1)_{n_1} T_{n_1} N_1 + (\mathrm{d}\phi_2)_{n_2} T_{n_2} N_2. \tag{5.1}$$

The maps ϕ_1 and ϕ_2 are then said to be *transversal* at m.

Notice in particular that if m is not in $|\phi_1|(|N_1|)$ or not in $|\phi_2|(|N_2|)$, then ϕ_1 and ϕ_2 are automatically transversal at m. Notice also that the sum in equation (5.1) needs not to be direct.

Proposition 5.3.12. *Let $\phi\colon L \to M$ be a supermanifold morphism and let (N, j) be a closed embedded submanifold of M, $\mathcal{O}(N) = \mathcal{O}(M)/\mathcal{J}_N$. Suppose that ϕ and j are transversal and denote by $\mathcal{J}$ the ideal in $\mathcal{O}(L)$ generated by $\phi^*(\mathcal{J}_N)$. Then $\mathcal{J}$ is a regular ideal corresponding to a closed submanifold $N' \subseteq L$.*

Proof. Let us first notice that $x \in |L|$ is such that $\mathcal{J} \subseteq \mathcal{J}_x$ if and only if $|\phi|(x) \in |N|$. Indeed $\phi^*(f)(x) = 0$ for all $f \in \mathcal{J}_N$ if and only if $f(|\phi|(x)) = 0$ for all $f \in \mathcal{J}_N$. If $x \in |N|$ then there exists a superchart $U, x^i, \bar{x}^j, \theta^r, \bar{\theta}^q$ such that $N \cap U$ is determined by $\bar{x}^j = \bar{\theta}^q = 0$. By using an appropriate global section, we can suppose that $\bar{x}^j, \bar{\theta}^q$ are defined on the whole M. We claim that $\phi^*(\bar{x}^j), \phi^*(\bar{\theta}^q)$ satisfy the properties of (1) of Proposition 5.3.5. They are clearly in $\mathcal{J}$ and they also generate $\mathcal{J}_x$. It remains only to prove that their differentials are linearly independent, but this is an easy consequence of the transversality condition. Indeed let $X_i \in T_x L$ be such that $(\mathrm{d}\phi)_x X_i = \left(\frac{\partial}{\partial \bar{x}^i}\right)_{|\phi|(x)}$. If $\sum c_i(\mathrm{d}\phi^*(\bar{x}^i)) = 0$ for some non-null sequence of numbers (c_i) then $\langle X_p, \sum c_i(\mathrm{d}\phi^*(\bar{x}^i))\rangle = \sum c_i \langle \phi_*(X_p), \bar{x}^i\rangle = \sum c_i \langle \left(\frac{\partial}{\partial \bar{x}^p}\right)_{|\phi|(x)}, x^i\rangle = 0$, which is clearly impossible. □

Proposition 5.3.13. *Let $\phi\colon L \to M$ be a supermanifold morphism and suppose that $m \in |M|$. Suppose that for each x in $|\phi|^{-1}(m)$ there exists a neighborhood where ϕ is a constant rank morphism.*

(1) *If $\mathcal{J}$ denotes the ideal in $\mathcal{O}(L)$ generated by $\phi^*(\mathcal{J}_m)$, then $\mathcal{J}$ distinguishes a closed supersubmanifold L' of L ($\mathcal{J}_m$ the ideal in $\mathcal{O}(M)$ corresponding to the point m).*

(2) *If (L', j) denotes the closed embedded submanifold distinguished by $\mathcal{J}$, then*

$$T_x L' \simeq \ker(\mathrm{d}\phi)_x$$

for each $x \in |L'| = |\phi|^{-1}(m)$.

Proof. (1) Suppose that $\mathcal{J} \subseteq \mathcal{J}_x$ for some $x \in |L|$. This means that, for all $f \in \mathcal{J}$, $f(x) = 0$ and since $f = \phi^*(g)$, with $g \in \mathcal{J}_m$, $g(|\phi|(x)) = 0$, in other words $x \in |\phi|^{-1}(m)$. Since ϕ has constant rank, for each $x \in |\phi|^{-1}(m)$ we can find coordinates $x^1, \dots, x^{r_1}, y^1, \dots, y^{r_2}, \xi^1, \dots, \xi^{s_1}, \eta^1, \dots, \eta^{s_2}$ around m such that $\phi^*(x^i), \phi^*(\xi^j)$ generate $\mathcal{J}_x$. The verification of (2) in Proposition 5.3.5 is an easy check. We thus have a closed embedded submanifold (L', j).

(2) Suppose now that $v \in T_x L'$ is given, i.e., a derivation $\mathcal{O}(L') \to \mathbb{R}$. Since, due to Proposition 5.3.4, $\mathcal{O}(L') \simeq \mathcal{O}(L)/I$, there is a bijective correspondence between

$T_x L'$ and derivations $v \in T_x L$ such that $\ker v \supseteq I$. Since I is the ideal generated by $\phi^*(\mathcal{J}_m)$, a simple check shows that $\ker v \supseteq I$ if and only if $(\mathrm{d}\phi)_x v = 0$, thus proving (2). □

Example 5.3.14. Consider the morphism $\phi_T\colon \mathrm{GL}_{m|n}(T) \to \mathbb{R}^{1|0}(T)$, $\phi(X) = \mathrm{Ber}(X)$, for T a supermanifold. This natural transformation between the functor of points of these two supermanifolds corresponds to the morphism of the superalgebras of global sections: $\phi^*\colon \mathcal{O}(\mathbb{R}^{1|0}) \to \mathcal{O}(\mathrm{GL}_{m|n})$, $\phi^*(t) = \mathrm{Ber}$, where t is the global canonical coordinate in $\mathbb{R}^{1|0}$ and Ber is the Berezinian function, that is, if x_{ij} and ξ_{kl} are the usual global canonical coordinates in $\mathcal{O}(\mathrm{GL}_{m|n})$, then

$$\mathrm{Ber} = \det(x_{kl})^{-1} \det(x_{ij} - \sum_{k,l} \xi_{ik} x^{kl} \xi_{lj}), \qquad \begin{array}{l} i,j = 1,\dots,m, \\ k,l = m+1,\dots,m+n, \end{array}$$

and x^{kl} denotes the element of the inverse of the matrix (x_{kl}). One can readily check that this is a submersion, hence a constant rank morphism at all points of $|\phi|^{-1}(1)$. By Proposition 5.3.13 we have that $|\phi|^{-1}(1)$ has a supermanifold structure and one can readily check that this is $\mathrm{SL}_{m|n}$, the supermanifold whose T-points are the matrices in $\mathrm{GL}_{m|n}(T)$ with Berezinian equal to 1.

The differential at the identity $(\mathrm{d}\phi)_{\mathrm{id}}$ can be calculated and it is the morphism

$$(\mathrm{d}\phi)_{\mathrm{id}}\colon T_{\mathrm{id}}(\mathrm{GL}_{m|n}) \cong \mathrm{M}_{m|n} \to T_1\mathbb{R}^{1|0} \cong \mathbb{R}, \quad M \mapsto \mathrm{str}(M),$$

where str denotes the supertrace of a matrix, i.e., $\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D)$. Hence the tangent space to $\mathrm{SL}_{m|n}$ at the identity consists of those matrices in $\mathrm{M}_{m|n}$, the real super vector space detailed in Example 3.1.5 with supertrace zero. We shall see in the next chapters that this super vector space has the extra structure of a super Lie algebra.

Using the method described in the previous example, one can easily calculate the tangent space at the identity for other classical supergroups that we shall define in the next chapter. However we shall give later a better tool to handle such calculations, namely the *stabilizer theorem*.

5.4 References

With respect to the classical material we refer the reader to [47], [1], [78] for an exhaustive treatment. Almost all the supergeometry material presented in this chapter is a reworking of [53]. More recent treatments of the same material can be found in [56], [76].

6

The Frobenius theorem

In ordinary geometry the Frobenius theorem is extremely important: it provides a constructive and effective way to build a submanifold of a given manifold starting from a subbundle of its tangent bundle, verifying some natural conditions. In this chapter we want to prove the extension to supergeometry of the Frobenius theorem, both in its local and global versions.

Though the results are stated in the super context almost unchanged, with respect to the ordinary setting, the arguments turn out to be more delicate and in particular, as we shall see, it is necessary to study separately the behaviour of odd vector fields and their associated distributions.

6.1 The local super Frobenius theorem

We want a mechanism by which we can construct locally a submanifold of a given supermanifold M.

Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold with tangent bundle Vec_M.

Definition 6.1.1. A *distribution* on M is an $\mathcal{O}_M$-submodule $\mathcal{D}$ of Vec_M which is locally a direct factor, namely, given $x \in |M|$, there exists an open neighbourhood U of x and another subsheaf $D' \subset \mathrm{Vec}_M$ so that

$$\mathrm{Vec}_{M,y} = \mathcal{D}_y \oplus \mathcal{D}'_y$$

for all $y \in U$.

We first want to show that any distribution $\mathcal{D}$ is locally free, in other words, $\mathcal{D}_x$ is a free $\mathcal{O}_x$-module. Before this, we need a lemma, whose proof relies heavily on the super version of Nakayama's lemma, detailed in Appendix B.

Lemma 6.1.2. *Let $x \in |M|$ and let $\{X_i, \Theta_j\}$ be vector fields defined in a neighbourhood of x and such that their associated tangent vectors are linearly independent. Then $X_{i,x}$, $\Theta_{j,x} \in \mathrm{Vec}_{M,x}$ are linearly independent.*

Proof. In Lemma B.3.3 take $E = \mathrm{span}_{\mathcal{O}_{M,x}}\{X_{i,x}, \Theta_{j,x}\}$ and $A^N = \mathrm{Vec}_{M,x}$. Then the result follows. □

Lemma 6.1.3. *Let $\mathcal{D}$ be as above. Then $\mathcal{D}_x$ is a free module for all $x \in |M|$. Moreover, if M is connected, then the dimension of the vector space $\mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x$ is independent from x, where $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_{M,x}$.*

Proof. Let $x \in |M|$. We set ourselves up to use the previous result, namely we will find vector fields $\{X_i, \Theta_j\}$ which generate $\mathcal{D}$ in an open neighborhood U of x so that the vectors corresponding to evaluating $\{X_i, \Theta_j\}$ at x are a basis for $T_x(M)$, the tangent space of M at x. Lemma 6.1.2 then tells us that $\{X_i, \Theta_j\}$ are in fact $\mathcal{O}_{M,x}$-linearly independent in $\mathrm{Vec}_{M,x}$.

Let $\{X_i, \Theta_j\}$ be vector fields on $\mathcal{D}_x$ defined on an open neighborhood U of x so that their tangent vectors at x are a basis for $\mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x \subseteq T_x(M)$. Because $\mathcal{D}$ is a distribution, it is locally free, and there exists $D \subseteq \mathrm{Vec}_M(U)$ so that $\mathrm{Vec}_{M,y} = \mathcal{D}_y \oplus D_y$ for all $y \in U$.

Now let $\{Y_k, \Xi_l\} \in D_x$ defined on U be such that their tangent vectors at x are a basis for $D_x/\mathfrak{m}_x D_x \subseteq T_x(M)$. Since $T_x(M) = \mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x \oplus D_x/\mathfrak{m}_x D_x$, it follows that $\{X_i, Y_k, \Theta_j, \Xi_l\}$ evaluated at x is a basis for $T_x(M)$. In fact, this is true for all points $y \in U$ by construction, hence by Lemma 6.1.2 they are linearly independent.

To see that $\{X_i, \Theta_j\}$ generate $\mathcal{D}$, let $\mathcal{D}'_y = \langle\{X_i, \Theta_j\}\rangle$ be the submodule of $\mathrm{Vec}_{M,y}$ generated by these vector fields. Similarly let $D'_y = \langle\{Y_k, \Xi_l\}\rangle$. Then $\mathcal{D}'_y \subseteq \mathcal{D}_y$ and $D'_y \subseteq D_y$, but we also know that $\mathcal{D}'_y \oplus D'_y = \mathrm{Vec}_{M,y} = \mathcal{D}_y \oplus D_y$, hence the generation.

This proves also that $\dim(\mathcal{D}_y/\mathfrak{m}_y\mathcal{D}_y)$ is invariant for $y \in U$, if M is connected. □

The next definitions are crucial for the statement of the local Frobenius theorem.

Definition 6.1.4. Let $M = (|M|, \mathcal{O}_M)$ be a connected supermanifold, that is, $|M|$ is connected, and let $\mathcal{D}$ be a distribution on M. We define the *rank* of a distribution as

$$\mathrm{rank}(\mathcal{D}) := \dim(\mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x).$$

The previous lemma ensures the rank is well defined (see also Appendix B, Section B.3).

Definition 6.1.5. We say that a distribution $\mathcal{D}$ is *involutive* if it is stable under the bracket on Vec_M, i.e., for vector fields X, Y in $\mathcal{D}$ the bracket $[X, Y]$ is also a vector field in $\mathcal{D}$.

Definition 6.1.6. We say that a distribution $\mathcal{D}$ is *integrable* if, for any $x \in |M|$, there exists (locally) a submanifold N of M whose tangent bundle can be identified locally with $\mathcal{D}$, that is,

$$\mathrm{Vec}_N|_U = \mathcal{D}|_U, \quad x \in U,$$

for a suitable neighbourhood U of x.

It is clear that an integral distribution is involutive; the Frobenius theorem in its local and global versions establishes a converse for this fact.

Lemma 6.1.3 and the definition of *rank* of a distribution allow us to make some crucial change of coordinates calculations at the coordinate chart level to shape an involutive distribution. We now prove a series of lemmas which will demonstrate, by construction, the local Frobenius theorem on supermanifolds.

Remark 6.1.7. The following lemmas pertain to the local Frobenius theorem and are local results, thus it suffices to consider the case $M = \mathbb{R}^{p|q}$ in a coordinate neighborhood of the origin, i.e., $U^{p|q}$.

Next is an explicit local calculation of the previous lemma.

Lemma 6.1.8. *Let $\mathcal{D}$ be an involutive distribution. Then there exist linearly independent supercommuting vector fields which span $\mathcal{D}$.*

Proof. Let $\{X_1, \ldots, X_r, \chi_1, \ldots, \chi_s\}$ be a basis for $\mathcal{D}$, in other words, $\{X_{1,x}, \ldots, X_{r,x}, \chi_{1,x}, \ldots, \chi_{s,x}\}$ form a basis for the free $\mathcal{O}_{M,x}$-module $\mathcal{D}_x$. Let $(t, \theta) = (t^1, \ldots, t^p, \theta^1, \ldots, \theta^q)$ be a local set of coordinates. Then we can express the vector fields

$$X_j = \sum_i a_{ji} \frac{\partial}{\partial t^i} + \sum_l \alpha_{jl} \frac{\partial}{\partial \theta^l}, \quad \chi_k = \sum_i \beta_{ki} \frac{\partial}{\partial t^i} + \sum_l b_{kl} \frac{\partial}{\partial \theta^l}. \tag{6.1}$$

The coefficients form an $r|s \times p|q$ matrix T,

$$T = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix},$$

with entries in $\mathcal{O}_M(U)$ (for a suitable domain U) and whose rows are the vector fields $X_1, \ldots, X_r, \chi_1, \ldots, \chi_s$ generating the distribution $\mathcal{D}$. T has rank $r|s$ since the $\{X_i, \chi_j\}$ are linearly independent. This is to say that the submatrix (a) has rank r and rank(b) = s. Then by renumerating coordinates (t, θ), we may assume that

$$T = (T_0|*),$$

where T_0 is an invertible $r|s \times r|s$ matrix. Multiplying T by any invertible matrix on the left does not change the row space of T (i.e., the distribution $\mathcal{D}$), so we can multiply by T_0^{-1} and assume that

$$T = \begin{pmatrix} I_r & 0 & * \\ 0 & I_s & * \end{pmatrix},$$

which amounts to saying that we may assume that

$$\begin{aligned} X_j &= \frac{\partial}{\partial t^j} + \sum_{i=r+1}^{p} a_{ji} \frac{\partial}{\partial t^i} + \sum_{l=s+1}^{q} \alpha_{jl} \frac{\partial}{\partial \theta^l}, \\ \chi_k &= \frac{\partial}{\partial \theta^k} + \sum_{l=s+1}^{q} b_{kl} \frac{\partial}{\partial \theta^l} + \sum_{i=r+1}^{p} \beta_{ji} \frac{\partial}{\partial t^i}. \end{aligned} \tag{6.2}$$

We then claim that $[X_j, X_k] = 0$. By the involutive property of $\mathcal{D}$, we know that

$$[X_j, X_k] = \sum_{i=1}^{r} f_i X_i + \sum_{l=1}^{s} \varphi_l, \chi_l$$

where the f_i are even functions and the φ_l are odd functions. Then, by (6.2), f_i is the coefficient of the $\frac{\partial}{\partial t^i}$ term in the vector field $[X_j, X_k]$. However, again by (6.2), it is clear that $[X_j, X_k]$ has only $\frac{\partial}{\partial t^i}$ terms for $i > r$, and so we have $f_i = 0$ for all i. Similarly, $[X_j, X_k]$ has only $\frac{\partial}{\partial \theta^l}$ terms for $l > s$, hence also $\varphi_l = 0$ for all l.

The cases $[X_j, \chi_k] = 0$ and $[\chi_l, \chi_k] = 0$ follow by using the same argument above. □

We now prove the local super Frobenius theorem in the case of an even distribution of rank $1|0$.

Lemma 6.1.9. *Let X be an even vector field. There exist local coordinates so that*

$$X = \frac{\partial}{\partial t^1}.$$

Proof. Let $\mathcal{J}$ be the ideal generated by the odd functions on $\mathbb{R}^{p|q}$. Then, since X is even, X maps $\mathcal{J}$ to itself. Thus X induces a vector field, and hence an integrable distribution, on the reduced space $\mathbb{R}^p$. Then we may apply the classical Frobenius theorem to get a coordinate system where $X = \frac{\partial}{\partial t^1}$ (mod $\mathcal{J}$). So we may assume that

$$X = \frac{\partial}{\partial t^1} + \sum_{i \geq 2} a_i \frac{\partial}{\partial t^i} + \sum_j \alpha_j \frac{\partial}{\partial \theta^j},$$

where the a_i are even, the α_j are odd, and $a_i, \alpha_j \in \mathcal{J}$. That the a_i are even implies that $a_i \in \mathcal{J}^2$. Moreover, we can find an even matrix (b_{jk}) so that $\alpha_j = \sum_k b_{jk}\theta^k$ (mod $\mathcal{J}^2$), and so modulo $\mathcal{J}^2$ we have

$$X = \frac{\partial}{\partial t^1} + \sum_{j,k} b_{jk}\theta^k \frac{\partial}{\partial \theta^j}.$$

Let $(t, \theta) \mapsto (y, \eta)$ be a change of coordinates where $y = t$ and $\eta = g(t)\theta$ for $g(t) = g_{ij}(t)$ a suitable invertible matrix of smooth functions depending on t_1 only, that is, $\eta^j = \sum_i g_{ji}(t)\theta^i$. Then

$$X = \frac{\partial}{\partial y^1} + \sum_{j,k} \theta^k \left(\frac{\partial g_{jk}}{\partial t^1} + \sum_l g_{jl} b_{lk} \right) \frac{\partial}{\partial \eta^j}, \tag{6.3}$$

and we choose $g(t)$ so that it satisfies the matrix differential equation and initial condition

$$\frac{\partial g}{\partial t^1} = -gb, \quad g(0) = I.$$

By (6.3) we may then assume that, modulo $\mathcal{J}^2$,

$$X = \frac{\partial}{\partial y^1}.$$

Next we claim that if $X = \frac{\partial}{\partial t^1} \pmod{\mathcal{J}^k}$, then $X = \frac{\partial}{\partial t^1} \pmod{\mathcal{J}^{k+1}}$. Since $\mathcal{J}$ is nilpotent, this claim will imply the result for the $1|0$-case.

Again, let $(t, \theta) \mapsto (y, \eta)$ be a change of coordinates so that $y^i = t^i + c_i$ and $\eta^j = \theta^j + \gamma_j$ for $c_i, \gamma_j \in \mathcal{J}^k$ suitably chosen in such a way that they depend on t_1 only. In the (t, θ) coordinate system, let

$$X = \frac{\partial}{\partial t^1} + \sum_{i \geq 2} h_i \frac{\partial}{\partial t^i} + \sum_u \varphi_u \frac{\partial}{\partial \theta^u}$$

for $h_i, \varphi_u \in \mathcal{J}^k$. In the new coordinate system, this becomes

$$X = \frac{\partial}{\partial y^1} + \sum_i \left(h_i + \frac{\partial c_i}{\partial t^1} \right) \frac{\partial}{\partial y^i} + \sum_l \left(\varphi_l + \frac{\partial \gamma_l}{\partial t^1} \right) \frac{\partial}{\partial \eta^l} + Y$$

for some $Y = 0 \pmod{\mathcal{J}^{k+1}}$ since $2k - 1 \geq k + 1$ for $k \geq 2$. So choose the c_i and γ_l so that they satisfy the differential equations

$$\frac{\partial c_i}{\partial t^1} = -h_i, \quad \frac{\partial \gamma_l}{\partial t^1} = -\varphi_l,$$

and we get that $X = \frac{\partial}{\partial y^1} \pmod{\mathcal{J}^{k+1}}$ as we wanted. □

The above Lemma 6.1.9 sets us up to prove the following.

Lemma 6.1.10. *Let $\mathcal{D}$ be a distribution generated by the set $\{X_1, \ldots, X_r\}$ of supercommuting even vector fields. Then there exist local coordinates (t, θ) so that*

$$X_j = \frac{\partial}{\partial t^j} + \sum_{i=1}^{j-1} a_{ij} \frac{\partial}{\partial t^i}$$

for some even functions a_{ij}.

Proof. We proceed by induction. The case $r = 1$ is presented above.

We may now assume that we can find coordinates which work for $r - 1$ supercommuting vector fields, and we want to prove the lemma for r. Again, assume that there are coordinates so that $X_j = \frac{\partial}{\partial t^j} + \sum_{i=1}^{j-1} a_{ij} \frac{\partial}{\partial t^i}$ for $j < r$. Then

$$X_r = \sum_{i=1}^{p} f_i \frac{\partial}{\partial t^i} + \sum_{k=1}^{q} \varphi_k \frac{\partial}{\partial \theta^k}$$

for some even functions f_i and odd functions φ_k. The assumption $[X_r, X_j] = 0$ gives

$$\sum f_i \left[\frac{\partial}{\partial t^i}, X_j \right] + \sum \varphi_k \left[\frac{\partial}{\partial \theta^k}, X_j \right] - \sum (X_j f_i) \frac{\partial}{\partial t^i} - \sum (X_j \varphi_k) \frac{\partial}{\partial \theta^k} = 0.$$

We know that $[\frac{\partial}{\partial t^i}, X_j]$ is a linear combination of $\frac{\partial}{\partial t^l}$ for $l < r$, which means that $X_j f_i = 0$ for all $j > r-1$. Because the coefficients of the X_j are "upper triangular" for $j \leq r-1$, we see that f_i depends only on $(t^r, \dots, t^p, \theta^1, \dots, \theta^q)$ for $i \geq r$. We also have $[\frac{\partial}{\partial \theta^k}, X_j] = 0$ for all k, and so $X_j \varphi_k = 0$ for all j as well. We can then again conclude that the φ_k depend only on $(t^r, \dots, t^p, \theta^1, \dots, \theta^q)$ as well.

Now we can rewrite X_r as follows:

$$X_r = \Big(\sum_{i=1}^{r-1} f_i \frac{\partial}{\partial t^i}\Big) + \underbrace{\sum_{l=r}^{p} f_l \frac{\partial}{\partial t^l} + \sum_{k=1}^{q} \varphi_k \frac{\partial}{\partial \theta^k}}_{= X_r'} .$$

Here X_r' depends only on $(t^r, \dots, \theta^q)$, and so by an application of Lemma 6.1.9 on X_r', we may change the coordinates $(t^r, \dots, \theta^q)$ so that $X_r' = \frac{\partial}{\partial t^r}$ and so

$$X_r = \frac{\partial}{\partial t^r} + \sum_{i=1}^{r-1} f_i' \frac{\partial}{\partial t^i}$$

(where the f_i' are the f_i above under the change of coordinates prescribed by Lemma 6.1.9). $\square$

In fact, the above lemma proves the local Frobenius theorem in the case when $\mathcal{D}$ is a purely even distribution (i.e., of rank $r|0$). For the most general case we need one more lemma, which establishes the local Frobenius theorem in the case of an odd distribution of rank $0|1$.

Lemma 6.1.11. *Say χ is an odd vector field so that $\chi^2 = 0$ and that* span$\{\chi\}$ *is a distribution. Then there exist coordinates so that locally $\chi = \frac{\partial}{\partial \theta^1}$.*

Proof. As we have previously remarked, since we want a local result, it suffices to assume that χ is a vector field on $\mathbb{R}^{p|q}$ near the origin. Let us say (y, η) are coordinates on $\mathbb{R}^{p|q}$. Then

$$\chi = \sum_i \alpha_i(y, \eta) \frac{\partial}{\partial y^i} + \sum_j a_j(y, \eta) \frac{\partial}{\partial \eta^j},$$

where the α_i are odd, the a_j are even, and we may assume that $a_1(0) \neq 0$.

Now consider the map

$$\pi \colon \mathbb{R}^{0|1} \times \mathbb{R}^{p|q-1} \to \mathbb{R}^{p|q}$$

given by $\pi^* \colon \mathcal{O}(\mathbb{R}^{p|q}) \to \mathcal{O}(\mathbb{R}^{0|1} \times \mathbb{R}^{p|q-1})$:

$$y^i = t^i + \epsilon \alpha_i(t, 0, \hat{\theta}), \quad \eta^1 = \epsilon a_1(t, 0, \hat{\theta}), \quad \eta^{j \geq 2} = \theta^j + \epsilon a_j(t, 0, \hat{\theta}).$$

Here ϵ is the coordinate on $\mathbb{R}^{0|1}$ and $(t^1, \dots, t^p, \theta^2, \dots, \theta^q)$ are the coordinates on $\mathbb{R}^{p|q-1}$, and $\hat{\theta}$ denotes the θ-indices $2, \dots, q$. The $\alpha(t, 0, \hat{\theta})$ and $a(t, 0, \hat{\theta})$ are the functions α_i and a_j where we substitute t for y, let $\theta^1 = 0$, and substitute $\hat{\theta}$ for $\eta^2, \ldots, \eta^q$. We claim that the map π is a diffeomorphism in a neighborhood of the origin. Indeed, the differential of π at 0 is

$$d\pi = \begin{pmatrix} I_p & * & 0 \\ 0 & a_1(0) & 0 \\ 0 & * & I_{q-1} \end{pmatrix}.$$

Since it is non-singular, we may think of $(t, \epsilon, \hat{\theta})$ as coordinates on $\mathbb{R}^{p|q}$ (near the origin) with π being a change of coordinates. Under this change of coordinates, we have

$$\frac{\partial}{\partial \epsilon} = \sum_i \alpha_i(t, 0, \hat{\theta}) \frac{\partial}{\partial y^i} + \sum_j a_j(t, 0, \hat{\theta}) \frac{\partial}{\partial \eta^j}.$$

The $\alpha_i(t, 0, \hat{\theta})$ and $a_j(t, 0, \hat{\theta})$ terms must be expressed as functions of (y, η). We shall do this by following the method we developed in the proof of the Chart Theorem 4.1.11. Notice that by a simple Taylor series expansion, $\alpha_i(y, \eta) = \alpha_i(t^i + \epsilon\alpha_i, \epsilon a_1, \theta^{k\geq 2} + \epsilon a_k) = \alpha_i(t^i, 0, \hat{\theta}) + \epsilon\beta_i$ for some even function β_i. Similarly we get $a_j(y, \eta) = a_j(t, 0, \hat{\theta}) + \epsilon b_j$ for some odd function b_j. Thus we can write

$$\frac{\partial}{\partial \epsilon} = \chi + \epsilon Z$$

for some even vector field Z. Recall that $\eta^1 = \epsilon \hat{a}_1$ where $\hat{a}_1$ is an even invertible section. Hence $\epsilon = \eta^1 A$ from some invertible even section A.

Then we see that under the change of coordinates given by π,

$$\frac{\partial}{\partial \epsilon} - \eta^1 \underbrace{A \cdot Z^*}_{=Z'} = \chi,$$

where Z^* denotes the pullback of Z by π and Z' is some even vector field (since both A and Z are even). Now,

$$\begin{aligned} \chi^2 = 0 &\implies (\tfrac{\partial}{\partial\epsilon} - \eta^1 Z')^2 = 0 \\ &\implies \underbrace{(\tfrac{\partial}{\partial\epsilon})^2}_{=0} - \tfrac{\partial}{\partial\epsilon}(\eta^1 Z') - (\eta^1 Z')\tfrac{\partial}{\partial\epsilon} + \underbrace{(\eta^1 Z')^2}_{=0} = 0 \\ &\implies -\hat{a}_1 Z' + \eta^1 \tfrac{\partial}{\partial\epsilon} Z' - \eta^1 Z' \tfrac{\partial}{\partial\epsilon} = 0 \\ &\implies \hat{a}_1 Z' = 0 \\ &\implies Z' = 0, \end{aligned}$$

so we really have $\frac{\partial}{\partial\epsilon} = \chi$ under the change of coordinates. □

Now we can prove the full local Frobenius theorem.

Theorem 6.1.12 (Local Frobenius Theorem). *Let $\mathcal{D}$ be an involutive distribution of rank $r|s$. Then there exist local coordinates so that $\mathcal{D}$ is spanned by*

$$\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^r}, \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^s}.$$

Proof. Let $\{X_1, \dots, X_r, \chi_1, \dots, \chi_s\}$ be a basis of vector fields for the distribution $\mathcal{D}$. By Lemma 6.1.8 we may assume that these basis elements supercommute, so then $\mathcal{D}' = \text{span}\{X_1, \dots, X_r\}$ is a subdistribution, and by Lemma 6.1.10 we get that there exist coordinates so that $X_i = \frac{\partial}{\partial t^i}$.

We then use the fact that $[\chi_1, X_i] = 0$ for all i to see that χ_1 depends only on coordinates $(t^{r+1}, \dots, \theta^q)$ (as in the proof of Lemma 6.1.10). In fact, this is not completely accurate. If we express χ_1 as in (6.1), we see that it is only the β_{ik} and b_{lk} which depend only on the coordinates $(t^{r+1}, \dots, \theta^q)$. However, we can always kill off the first r $\partial/\partial t^i$ terms by subtracting appropriate linear combinations of the $\{X_1 = \partial/\partial t^1, \dots, X_r = \partial/\partial t^r\}$.

Since $\chi_1^2 = 0$, by Lemma 6.1.11, we may change only the coordinates $(t^{r+1}, \dots, \theta^q)$ and express $\chi_1 = \frac{\partial}{\partial \theta^1}$. For χ_2 we apply the same idea: that $[\chi_2, X_i] = 0$ and $[\chi_2, \chi_1] = 0$ again shows that χ_2 depends only on coordinates $(t^{r+1}, \dots, t^p, \theta^2, \dots, \theta^q)$, and applying Lemma 6.1.11 once again gives $\chi_2 = \frac{\partial}{\partial \theta^2}$. The same argument then applies successively to $\chi_3, \dots, \chi_s$. □

We are now in a position to state and prove the global Frobenius theorem on supermanifolds.

6.2 The global super Frobenius theorem

Theorem 6.2.1 (Global Frobenius theorem). *Let M be a supermanifold, and let $\mathcal{D}$ be an involutive distribution on M. Then given any point of M there is a unique maximal supermanifold corresponding to $\mathcal{D}$ which contains that point; in other words, $\mathcal{D}$ is integrable.*

Proof. Let $\mathcal{D} = \text{span}\{X_1, \dots, X_r, \chi_1, \dots, \chi_s\}$ as in the previous section (again the X_i are even and the χ_j are odd). Let $\mathcal{D}_0 = \text{span}\{X_1, \dots, X_r\}$; this subdistribution maps odd sections to odd sections, and so descends to an integral distribution $\widetilde{\mathcal{D}}_0$ on $\widetilde{M}$. Let $x \in |M|$. Then by the classical global Frobenius Theorem, there is a unique maximal integral manifold $\widetilde{M}_x \subset \widetilde{M}$ of $\widetilde{\mathcal{D}}_0$ containing x. We want to build a sheaf of commutative superalgebras on $\widetilde{M}_x$.

By the local Frobenius theorem, given any point $y \in |M|$, there exists an open coordinate neighborhood around y, $U_y \subset |M|$, so that U_y is characterized by coordinates (t, z, θ, η) (i.e., $\mathcal{O}_M(U_y) = C^\infty(t, z)[\theta, \eta]$), where $\mathcal{D}$ is given by the $\mathcal{T}_M$-span of $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\}$. Now let $U \subset |M|$ and define the following presheaf $\mathcal{I}$ on $|M|$:

$$\mathcal{I}(U) = \langle \{ f \in \mathcal{O}_M(U) \mid \text{for all } y \in \widetilde{M}_x \cap U \text{ there exists } V_y \subset U \text{ such that } f|_{V_y} \in C^\infty(z)[\eta] \} \rangle.$$

We claim that $\mathcal{I}$ is a subsheaf of $\mathcal{O}_M$. Again let $U \subset |M|$ be an open subset and let $\{U_\alpha\}$ be an open covering of U so that for a family of $s_\alpha \in \mathcal{I}(U_\alpha)$ we have $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$. We know that there exists a unique $s \in \mathcal{O}_M(U)$ so that $s|_{U_\alpha} = s_\alpha$. Let $y \in M_x \cap U$. Then $y \in U_\alpha$ for some α. There exists $V_y \subset U_\alpha$ where $s|_{V_y} \in C^\infty(z)[\eta]$ since $s|_{V_y} = s_\alpha|_{V_y}$. Hence $\mathcal{I}$ is a subsheaf of $\mathcal{O}_M$. Moreover, $\mathcal{I}$ is an ideal sheaf by construction.

It is clear that if $p \notin \widetilde{M}_x$, then $\mathcal{I}_p = \mathcal{O}_{M,p}$ since we can find some neighborhood of p, $W_p \cap \widetilde{M}_x = \emptyset$, where $\mathcal{I}(W_p) = \mathcal{O}_M(W_p)$. Thus the support of $\mathcal{I}$ is $\widetilde{M}_x$, and we have specified a sheaf of ideals with support $\widetilde{M}_x$ which defines a unique closed submanifold of M (see Section 5.3). By going to coordinate neighborhoods it is clear that this closed subspace is in fact a closed subsupermanifold which we shall now call M_x.

The maximality condition is clear. From the classical theory we have that the reduced space is maximal, and locally we can verify that we have the maximal number of odd coordinates that $\mathcal{D}$ allows. □

6.3 References

The local Frobenius theorem for even distributions was proved by Giachetti–Ricci in [35], while the version for odd distributions was considered in [50] by Koszul. The first proof of the local and global super Frobenius theorem we are aware of is the one contained in [19]. For other more modern treatments we refer the reader to [56], [22], [76].

7

Super Lie groups

In this chapter we want to take a closer look at supermanifolds with a group structure: *Lie supergroups* or *super Lie groups*. As in the ordinary setting, a super Lie group is defined as a supermanifold together with the multiplication and inverse morphisms, that satisfy the usual properties expressed in terms of certain commutative diagrams. To any Lie supergroup, we can naturally associate a Lie superalgebra, consisting of the left-invariant vector fields, As in the ordinary setting, the Lie superalgebra is identified with the tangent superspace to the supergroup at the identity.

We can equivalently approach this theory using the language of the *super Harish-Chandra pairs* (SHCP for short), which is historically how it was originally developed by Kostant and Koszul [49], [50]. The SHCP allows us to recover the supergroup structure of a Lie supergroup G only by knowing its reduced ordinary Lie group $\widetilde{G}$ and its associated Lie superalgebra $\mathfrak{g}$. In fact an even stronger statement is true: there is an equivalence of categories between the SHCP and the super Lie groups. As we are going to see in the next chapter, this equivalence extends to the categories of actions of the SHCP and the super Lie groups, respectively.

7.1 Super Lie groups

A Lie group is a group object in the category of manifolds. Likewise a super Lie group is a group object in the category of supermanifolds. More precisely:

Definition 7.1.1. A real *super Lie group* G (SLG for short) is a real smooth super manifold G together with three morphisms

$$\mu\colon G\times G\to G,\quad i: G\to G,\quad e\colon \mathbb{R}^{0|0}\to G,$$

called *multiplication*, *inverse*, and *unit* respectively satisfying the commutative diagrams

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\mu\times\mathrm{id}} & G\times G \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\mu} & & \Big\downarrow{\scriptstyle \mu} \\ G\times G & \xrightarrow[\mu]{} & G, \end{array}$$

$$\begin{array}{ccccc} & & G\times G & & \\ & {\scriptstyle \langle \mathrm{id}_G,\hat{e}\rangle}\nearrow & & \searrow{\scriptstyle \mu} & \\ G & & \xrightarrow{\mathrm{id}_G} & & G, \\ & {\scriptstyle \langle \hat{e},\mathrm{id}_G\rangle}\searrow & & \nearrow{\scriptstyle \mu} & \\ & & G\times G & & \end{array} \qquad \begin{array}{ccccc} & & G\times G & & \\ & {\scriptstyle \langle \mathrm{id}_G,i\rangle}\nearrow & & \searrow{\scriptstyle \mu} & \\ G & & \xrightarrow{\hat{e}} & & G, \\ & {\scriptstyle \langle i,\mathrm{id}_G\rangle}\searrow & & \nearrow{\scriptstyle \mu} & \\ & & G\times G & & \end{array}$$

where $\hat{e}$ denotes the composition of the identity $e\colon \mathbb{R}^{0|0} \to G$ with the unique map $G \to \mathbb{R}^{0|0}$. Moreover, $\langle \psi, \phi \rangle$ denotes the map $(\psi \times \phi) \circ \mathrm{d}_G$, $\mathrm{d}_G\colon G \to G \times G$ being the canonical diagonal map.

We may of course interpret all these maps and diagrams in the language of T-points, which gives us (for any supermanifold T) morphisms $\mu_T\colon G(T) \times G(T) \to G(T)$, etc. that obey the same commutative diagrams. In other words, if G is an SLG then the functor $T \to G(T) = \mathrm{Hom}(T, G)$ takes values in the category of set theoretical groups. Conversely, Yoneda's lemma says that if the functor $T \to G(T)$ takes values in the category of set theoretical groups, then G is actually a super Lie group. This leads us to an alternative definition of a super Lie group.

Definition 7.1.2. A supermanifold G is a *super Lie group* if for any supermanifold T, $G(T)$ is a group, and for any supermanifold S and morphism $T \to S$, the corresponding map $G(S) \to G(T)$ is a group homomorphism.

In other words, a supermanifold G is a super Lie group if and only if its functor of points $T \mapsto G(T)$ is a functor into the category of groups.

Remark 7.1.3. Let us notice that to each super Lie group G is associated a Lie group $\widetilde{G}$. It is defined as the underlying manifold $\widetilde{G}$ with the "reduced morphisms"

$$|\mu|\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}, \quad |i|\colon \widetilde{G} \to \widetilde{G}, \quad |e|\colon \mathbb{R}^0 \to \widetilde{G}.$$

Since the map $\phi \mapsto |\phi|$ that associates to any supermanifold morphism $\phi\colon M \to N$ the morphism $|\phi|\colon \widetilde{M} \to \widetilde{N}$ between the associated reduced manifolds is functorial, it is immediate that $(\widetilde{G}, |\mu|, |i|, |e|)$ is a Lie group. Notice also that in the ordinary setting the reduced morphisms $|\mu|$, $|i|$ and $|e|$ fully determine the Lie group structure on the manifold $\widetilde{G}$.

Example 7.1.4. Let us consider the super Lie group $\mathbb{R}^{1|1}$ through the symbolic language of T-points. The product morphism $\mu\colon \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ is given by

$$(t, \theta) \cdot (t', \theta') = (t + t' + \theta\theta', \theta + \theta') \tag{7.1}$$

where the coordinates (t, θ) and (t', θ') represent two distinct T-points for some supermanifold T. It is then clear by the formula (7.1) that the group axioms are satisfied. We are going to return with more details in Example 7.2.4.

Remark 7.1.5. Notice that the properties required in Definition 7.1.1 translate into properties of the morphisms on the global sections, $\mu^*\colon \mathcal{O}(G) \to \mathcal{O}(G \times G)$, $i^*\colon \mathcal{O}(G) \to \mathcal{O}(G)$, that make $\mathcal{O}(G)$ "almost" a Hopf superalgebra (see Chapter 1, Section 1.7). One word of caution: since $\mathcal{O}(G \times G) \cong \mathcal{O}(G)\hat{\otimes}\mathcal{O}(G)$, strictly speaking $\mathcal{O}(G)$ is not a Hopf superalgebra but a *topological Hopf superalgebra*, meaning that since we are taking a completion of the tensor product, we are allowed to consider infinite sums (for a definition of topological Hopf algebras see [20]). In fact an SLG can also be equivalently defined as a supermanifold G, where $\mathcal{O}(G)$ has a topological Hopf superalgebra structure. The product in $G(T)$ is recovered via the comultiplication μ^* in the following way:

$$x \cdot y = m_{\mathcal{O}(T)} \circ x \times y \circ \mu^*, \quad x, y \in G(T) = \mathrm{Hom}(\mathcal{O}(G), \mathcal{O}(T)).$$

Notice that we are making implicit use of Proposition 4.5.4, which tells us that specifying a morphism on $\mathcal{O}(G) \otimes \mathcal{O}(G)$ uniquely determines it on its completion $\mathcal{O}(G \times G)$.

We leave to the reader the easy check of the fact that the properties of multiplication μ, unit e and inverse i of G will correspond to the (dual) properties of μ^*, e^*, i^* that make them respectively comultiplication, counit and antipode in the topological Hopf superalgebra sense. We are going to come back to this with more details when we talk about the algebraic setting in Chapter 11, where completion is no longer necessary.

In the language of T-points, as we already remarked, the definition of Lie supergroup is equivalent to saying that a super Lie group as a functor from the category of supermanifolds to the category of groups is representable. In this vein, let us further examine the $\mathrm{GL}_{m|n}$ example.

Example 7.1.6. As one can readily see, the supermanifold $\mathrm{GL}_{m|n}$ described in Example 4.6.3 (2) is a Lie supergroup. In fact $\mathrm{GL}_{m|n}(T)$ is the group of automorphisms of the $\mathcal{O}(T)_0$-module $\mathbb{R}^{m|n}(T) = (\mathcal{O}(T) \times \mathbb{R}^{m|n})_0$.

Let us now consider another example of a super Lie group, $\mathrm{SL}_{m|n}$. We define $\mathrm{SL}_{m|n}$ in a way that mimics the classical construction. For each supermanifold T, the Berezinian gives a morphism from the T-points of $\mathrm{GL}_{m|n}$ to the T-points of $\mathrm{GL}_{1|0}$:

$$\mathrm{Ber}_T\colon\ \mathrm{GL}_{m|n}(T) \to \mathrm{GL}_{1|0}(T), \quad X \mapsto \mathrm{Ber}(X).$$

The super special linear group $\mathrm{SL}_{m|n}$ is the kernel of Ber_T. Certainly this functor is group-valued, however it is not immediately clear that it is the functor of points of a supermanifold, in other words that it is representable. We have already discussed the representability in the previous chapter, however we are going to address this issue more generally in the next chapter, when we discuss the *Stabilizer Theorem* that will give us the representability of this and the other functor of points of the classical supergroups.

7.2 The super Lie algebra of a super Lie group

For an ordinary Lie group H, we can define a morphism ℓ_h, the left multiplication by $h \in H$, as

$$H \xrightarrow{\ell_h} H, \quad a \mapsto ha$$

(for $a \in H$). The differential of this morphism gives

$$T_a(H) \xrightarrow{(d\ell_h)_a} T_{ha}(H)$$

and for a vector field X on H, we say that X is *left-invariant* if

$$d\ell_h \circ X = X \circ \ell_h.$$

We want to interpret this in the super category by saying that a left-invariant vector field on G is invariant with respect to the group law μ^* "on the left".

Let G be a super Lie group with group law $\mu\colon G \times G \to G$ and let us denote by $\mathbb{1}$ the identity at the level of sheaf morphisms.

Let Vec_G denote the vector fields on G (see Chapter 4, Section 4.4).

Definition 7.2.1. A vector field $X \in \mathrm{Vec}_G$ is said to be *left-invariant* if

$$(\mathbb{1} \otimes X) \circ \mu^* = \mu^* \circ X.$$

Similarly a vector field $X \in \mathrm{Vec}_G$ is said to be *right-invariant* if

$$(X \otimes \mathbb{1}) \circ \mu^* = \mu^* \circ X.$$

Since the bracket of left-invariant vector fields is left-invariant, as one can readily check, the left-invariant vector fields are a super Lie subalgebra of Vec_G, which we denote by $\mathfrak{g}$.

Definition 7.2.2. Let G be a super Lie group. Then

$$\mathfrak{g} = \{X \in \mathrm{Vec}_G \mid (\mathbb{1} \otimes X)\mu^* = \mu^* X\}$$

is the *super Lie algebra* associated with the super Lie group G, and we write $\mathfrak{g} = \mathrm{Lie}(G)$ as usual.

The next proposition says that $\mathfrak{g} = \mathrm{Lie}(G)$ is a finite-dimensional super vector space canonically identified with the super tangent space at the identity of the super Lie group G.

Proposition 7.2.3. *Let G be a super Lie group.*

(1) *If X_e denotes a vector in T_eG, then*

$$X := (\mathbb{1} \otimes X_e)\mu^*$$

is a left-invariant vector field. Similarly $X^R := (X_e \otimes \mathbb{1})\mu^*$ is a right-invariant vector field.

(2) *The map*

$$T_eG \to \mathfrak{g}, \quad X_e \to X := (\mathbb{1} \otimes X_e)\mu^*, \tag{7.2}$$

is an isomorphism of super vector spaces. Similarly for right vector fields.

Proof. To prove (1) for the left-invariant vector fields, we need to show that

$$[\mathbb{1} \otimes ((\mathbb{1} \otimes X_e)\mu^*)] \circ \mu^* = \mu^* \circ [(\mathbb{1} \otimes X_e)\mu^*].$$

This is a simple check that uses the coassociativity of μ^*, that is, $(\mathbb{1} \otimes \mu^*)\mu^* = (\mu^* \otimes \mathbb{1})\mu^*$. In fact

$$\begin{aligned}[\mathbb{1} \otimes (\mathbb{1} \otimes X_e) \circ \mu^*] \circ \mu^* &= (\mathbb{1} \otimes \mathbb{1} \otimes X_e)[(\mathbb{1} \otimes \mu^*) \circ \mu^*] \\ &= (\mathbb{1} \otimes \mathbb{1} \otimes X_e)[(\mu^* \otimes \mathbb{1}) \circ \mu^*] \\ &= \mu^* \circ [(\mathbb{1} \otimes X_e)\mu^*].\end{aligned}$$

As for (2) we notice that the injectivity of the map (7.2) is immediate. Let us thus focus on the surjectivity. Suppose that X is a left-invariant vector field, i.e., $(\mathbb{1} \otimes X)\mu^* = \mu^* X$. Apply $\mathbb{1} \otimes e^*$ to this equality to obtain

$$(\mathbb{1} \otimes e^*)(\mathbb{1} \otimes X)\mu^* = (\mathbb{1} \otimes e^*)\mu^* X,$$

from which we get $X = (\mathbb{1} \otimes X_e)\mu^*$ since $(\mathbb{1} \otimes e^*)\mu^* = \mathbb{1}$. So we are done. □

We can use the previous proposition to endow T_eG with the structure of a super Lie algebra and to identify it with $\mathfrak{g}$. From now on we shall use such an identification freely without an explicit mention.

Example 7.2.4. We want to calculate the left-invariant vector fields on $\mathbb{R}^{1|1}$ by means of the group law (7.1),

$$(t, \theta) \cdot_\mu (t', \theta') = (t + t' + \theta\theta', \theta + \theta'),$$

from Example 7.1.4. In terms of $\mu^* \colon \mathcal{O}(\mathbb{R}^{1|1}) \to \mathcal{O}(\mathbb{R}^{1|1} \times \mathbb{R}^{1|1})$ the group law reads

$$\mu^*(t) = t \otimes \mathbb{1} + \mathbb{1} \otimes t + \theta \otimes \theta, \quad \mu^*(\theta) = \theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta.$$

From Proposition 7.2.3, we know that the Lie algebra of left-invariant vector fields can be extracted from $T_eG = \operatorname{span}\left\{\frac{\partial}{\partial t}|_e, \frac{\partial}{\partial \theta}|_e\right\}$. We use the identity (7.2) to calculate the corresponding left-invariant vector fields:

$$\left(\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e\right) \circ \mu^*, \quad \left(\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e\right) \circ \mu^*. \tag{7.3}$$

To get coordinate representations of (7.3), we apply them to coordinates (t, θ):

$$\begin{aligned}(\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e) \circ \mu^*(t) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e)(t \otimes \mathbb{1} + \mathbb{1} \otimes t + \theta \otimes \theta) = 1, \\ (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e) \circ \mu^*(\theta) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e)(\theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta) = 0, \\ (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e) \circ \mu^*(t) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e)(t \otimes \mathbb{1} + \mathbb{1} \otimes t + \theta \otimes \theta) = -\theta, \\ (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e) \circ \mu^*(\theta) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e)(\theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta) = 1.\end{aligned}$$

Thus the left-invariant vector fields on $(\mathbb{R}^{1|1}, \mu)$ are

$$X = \frac{\partial}{\partial t}, \quad Y = -\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}. \tag{7.4}$$

A quick check using the definition shows that (7.4) are in fact left-invariant. The Lie superalgebra structure can now be easily computed: $\mathrm{Lie}(\mathbb{R}^{1|1}) = \mathrm{span}\{X, Y\}$ with brackets

$$[X, X] = 0, \quad [X, Y] = 0, \quad [Y, Y] = -2X.$$

Proposition 7.2.5. *Let G and H be super Lie groups and let $\phi\colon G \to H$ be a morphism of super Lie groups. The map*

$$(\mathrm{d}\phi)_e \colon \mathfrak{g} \to \mathfrak{h}$$

is a super Lie algebra homomorphism.

Proof. The only thing to check is that $(\mathrm{d}\phi)_e$ preserves the super Lie bracket. We leave this to the reader as an easy exercise, recalling that the bracket has always to be computed on the left-invariant vector fields. □

Corollary 7.2.6. *The even part of the super Lie algebra* $\mathrm{Lie}(G)$ *canonically identifies with the ordinary Lie algebra* $\mathrm{Lie}(\widetilde{G})$.

Proof. This is immediate considering the canonical inclusion $j\colon \widetilde{G} \to G$ and the previous proposition. □

We end this section showing that the reduced Lie group $\widetilde{G}$ acts on G in a natural way.

Definition 7.2.7. If G is a super Lie group and M is a supermanifold, we say that G *acts* on M if we have a morphism

$$a\colon G \times M \to M$$

defined using the functor of points as

$$a_T\colon G(T) \times M(T) \to M(T), \quad (g, m) \mapsto g \cdot m,$$

such that

(1) $e_T \cdot m = m$,

(2) $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$.

In other words,

$$a \circ \langle \hat{e}, \mathbb{1}_M \rangle = \mathbb{1}_M, \tag{7.5a}$$

$$a \circ (\mu \times \mathbb{1}_M) = a \circ (\mathbb{1}_G \times a), \tag{7.5b}$$

where $\mathbb{1}_M : M \to M$ denotes the identity morphism of a supermanifold M and $\hat{e} \colon G \to G$ is a super morphism defined, in the functor of points notation, as $\hat{e}_T(g) = e_T$, with e_T the identity element in the group $G(T)$.

The morphism a is called an *action* of G on M. We will return more extensively to actions in Chapter 8.

Remark 7.2.8. We will encounter also *right actions* (as opposed to the previous ones that are called *left actions*). For them, condition (2) is replaced by $g_1 \cdot (g_2 \cdot m) = (g_2 g_1) \cdot m$, while condition (7.5b) is replaced by

$$a \circ (\mu \times \mathbb{1}_M) = a \circ (\mathbb{1}_G \times a) \circ (\tau \times \mathbb{1}_M),$$

where $\tau \colon G \times G \to G$ is the flip morphism defined by $\tau_T(g_1, g_2) = g_2 g_1$ for $g_i \in G(T)$.

Since G is a super Lie group, it acts on itself via group multiplication $\mu \colon G \times G \to G$. Fix an element $\bar{g} \colon \mathbb{R}^{0|0} \to G$ in $|G|$ (recall that a topological point is identified with an element in $G(\mathbb{R}^{0|0})$, see Example 3.4.6 (i)). Moreover a topological point $\bar{g}$ can always be viewed naturally as a T-point of G. Define the *left translation* by $\bar{g}$:

$$l_{\bar{g}} \colon G \simeq \mathbb{R}^{0|0} \times G \xrightarrow{\bar{g} \times \mathrm{id}} G \times G \xrightarrow{\mu} G.$$

This induces an action

$$\underline{a} \colon \widetilde{G} \times G \to G,$$

which we call left multiplication by $\widetilde{G}$. At the level of sections we have

$$l_{\bar{g}}^* f = (\mathrm{ev}_{\bar{g}} \otimes \mathbb{1}) \circ \mu^*(f), \quad \underline{a}^* f = (j \otimes \mathbb{1}) \circ \mu^*(f),$$

where j denotes the embedding of $\widetilde{G}$ in G.

Similarly we can define the *right translation* by $g \in |G|$ by

$$r_{\bar{g}}^* f = (\mathbb{1} \otimes \mathrm{ev}_{\bar{g}}) \circ \mu^*(f).$$

The corresponding morphism $G \times \widetilde{G} \to G$ defines a right action.

7.3 The Hopf superalgebra of distributions

In this section we are going to briefly describe Kostant's original approach to the theory of supergroups and we are also going to prove that this is essentially equivalent to our treatment.

As we have seen in Section 4.7, if M is a supermanifold, the superalgebra of distributions with finite support $\mathcal{O}(M)^\circ$ is a super coalgebra. We can use the group structure of G to define a superalgebra structure on $\mathcal{O}(G)^\circ$, which is compatible with its coalgebra structure. As we shall see, the presence of an antipode makes $\mathcal{O}(G)^\circ$ a Hopf superalgebra, not to be confused with the topological Hopf superalgebra $\mathcal{O}(G)$ described above.

Definition 7.3.1. Let X_g, $Y_{\bar{g}}$ be point supported distributions (resp. supported at the points g and $\bar{g}$) on the Lie supergroup G. Define the *convolution product* of X_g and $Y_{\bar{g}}$ by

$$(X_g * Y_{\bar{g}})(f) = \langle X_g \otimes Y_{\bar{g}}, \mu^*(f) \rangle,$$

where $f \in \mathcal{O}(G)$ and μ is the super Lie group multiplication. Given a distribution X_g we also define

$$(\mathrm{d}l_{\bar{g}})_g X_g = X_g \circ l_{\bar{g}}^*.$$

Despite the analogy in notation and properties, distributions must not be confused with vector fields. We shall establish how distributions and vector fields are related later in this section.

Remark 7.3.2. Notice that the convolution product $X_g * Y_{\bar{g}}$ of the distributions X_g and $Y_{\bar{g}}$, supported at g and $\bar{g}$, respectively, is supported at $g \cdot \bar{g}$. This is an immediate consequence of the statement for ordinary convolutions, that can be found for example in [26], Ch. XVII, n. 11.

The next proposition provides us with a generalization of well-known formulas for distributions in the ordinary setting.

Proposition 7.3.3. *Let X_g be a distribution on a Lie supergroup.*

(1) $(\mathrm{d}l_{\bar{g}})_g X_g = \mathrm{ev}_{\bar{g}} * X_g$.

(2) $(\mathrm{d}r_{\bar{g}})_g X_g = X_g * \mathrm{ev}_{\bar{g}}$.

Proof. We shall prove only (1), (2) being the same.

$$\begin{aligned}\langle (\mathrm{d}l_{\bar{g}})_g X_g, f \rangle &= \langle X_g, l_{\bar{g}}^* f \rangle \\ &= \langle X_g, (\mathrm{ev}_{\bar{g}} \otimes \mathbb{1}) \circ \mu^*(f) \rangle \\ &= (\mathrm{ev}_{\bar{g}} * X_g)(f).\end{aligned}$$

□

The above discussion establishes the following proposition.

Proposition 7.3.4. *The convolution product defines a superalgebra structure on the supercoalgebra $\mathcal{O}(G)^\circ$ of distributions with finite support.*

Proof. By its very definition $*$ is associative and, by Proposition 7.3.3, ev_e is its identity element. □

We now come to a closer examination of the algebraic structure of $\mathcal{O}(G)^\circ$. We know that $\mathcal{O}(G)^\circ$ is a super-coalgebra from Section 4.7 and a superalgebra with identity ev_e from the previous proposition. It is only natural to ask whether the two structures are compatible.

Proposition 7.3.5. *$\mathcal{O}(G)^\circ$ is a super Hopf algebra with comultiplication Δ, counit η and antipode S given by*

$$\langle \Delta(X_g), f \otimes g\rangle := \langle X_g, f \cdot g\rangle, \quad \langle \eta(X_g), f\rangle := \langle X_g, \mathrm{ev}_e(f)\rangle,$$
$$\langle S(X_g), f\rangle := \langle X_g, i^*(f)\rangle,$$

where $i\colon G \to G$ denotes the inverse morphism.

Proof. First we show that the comultiplication Δ and the counit η are superalgebra morphisms. Let us start with the comultiplication Δ. Consider

$$\begin{aligned} \langle \Delta(X_g * Y_h), s \otimes t\rangle &= \langle X_g \otimes Y_h, \mu^*(s \cdot t)\rangle \\ &= \langle X_g \otimes Y_h, \mu^*(s)\mu^*(t)\rangle \\ &= \langle \Delta X_g * \Delta Y_h, s \otimes t\rangle. \end{aligned}$$

Consider now the counit η. We have

$$\begin{aligned} \langle \eta(X_g * Y_h), s\rangle &= \langle X_g \otimes Y_h, \mu^* \circ \mathrm{ev}_e(s)\rangle \\ &= \langle X_g \otimes Y_h, \mathrm{ev}_e(s) 1 \otimes 1\rangle \\ &= \mathrm{ev}_e(s) X_g(1) Y_h(1) \\ &= \langle \eta(X_g) * \eta(Y_h), s\rangle. \end{aligned}$$

The fact that S has the property of the antipode is a similar check. □

We end this section by pointing out an interesting decomposition of $\mathcal{O}(G)^\circ$ which parallels a general structure theorem for Hopf algebras (see again [72] for the classical case).

Let us examine more closely the product law in $\mathcal{O}(G)^\circ$. Fix a distribution X_g in $\mathcal{O}(G)^\circ$. Each such element can be canonically written as the convolution product of a distribution at the identity e of the super Lie group with the "element" g of $|G|$:

$$X_g = \mathrm{ev}_g * (\mathrm{ev}_{g^{-1}} * X_g) = \mathrm{ev}_g * (\mathrm{d}l_{g^{-1}})_g X_g. \tag{7.6}$$

Thus each element in $\mathcal{O}(G)^\circ$ can be written as a product $\mathrm{ev}_g * X_e$, where $X_e \in \mathcal{O}(G)^\circ$ denotes the distribution at e given by

$$X_e := (\mathrm{d}l_{g^{-1}})_g X_g.$$

Notice that here X_e is used in a more general sense than in Section 11.2.4, where it denoted an element in $T_e G$.

The next proposition characterizes the distributions with support at e.

Proposition 7.3.6. *The space of distributions supported at the identity e of the super Lie group G is isomorphic, as a super Hopf algebra, to the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the super Lie algebra $\mathfrak{g}$ associated with the supergroup G.*

Proof. It goes exactly as in the classical case, but we shall nevertheless briefly outline it to make the text self-contained. As we have seen in Chapter 4, Section 4.7, the finitely supported distributions can be identified with differential operators. Hence a distribution X_e supported at the identity can be identified with a derivation of type

$$X_e = \sum_{I,J} c_{I,J} \frac{\partial}{\partial x^I}\bigg|_e \frac{\partial}{\partial \xi^J}\bigg|_e,$$

where (x_i, ξ_j) are local coordinates around the identity $e \in |G|$ and I and J are multi-indices. Every distribution X_e described above corresponds to a unique left-invariant differential operator via the linear map

$$X_e \mapsto (\mathbb{1} \otimes X_e)\mu^*,$$

very much in the same way as tangent vectors at the identity correspond to left-invariant vector fields. The linear map described above is a linear isomorphism; the proof of this fact resembles very closely what we did in Proposition 7.2.3. Finally a left-invariant differential operator can be uniquely identified with an element of the universal enveloping superalgebra of $\mathfrak{g}$, a conclusion from dimension considerations and using the PBW theorem. □

Using decomposition (7.6), we can now define a linear map

$$\psi : \mathcal{O}(G)^\circ \to \mathbb{R}\widetilde{G} \otimes \mathfrak{U}(\mathfrak{g}), \quad X_g \mapsto g \otimes (\mathrm{d}l_{g^{-1}})_g X_g,$$

where we notice that $(\mathrm{d}l_{g^{-1}})_g X_g$ is in fact a distribution supported at the identity, hence identified with an element in $\mathfrak{U}(\mathfrak{g})$ (see previous proposition). $\mathbb{R}\widetilde{G}$ denotes the group algebra of $\widetilde{G}$ and consists of the formal finite sums of elements in $\widetilde{G}$ with coefficients in $\mathbb{R}$. By Proposition 7.3.6 and equation (7.6), the above map is a bijection.

All the algebraic structures defined over $\mathcal{O}(G)^\circ$ can thus be transported to corresponding structures over $\mathbb{R}\widetilde{G} \otimes \mathfrak{U}(\mathfrak{g})$, as the next proposition formalizes.

Proposition 7.3.7. $\mathbb{R}\widetilde{G} \otimes \mathfrak{U}(\mathfrak{g})$ *is a super Hopf algebra. The coalgebra structure is the one induced by the coalgebra structures of* $\mathbb{R}\widetilde{G}$ *and* $\mathfrak{U}(\mathfrak{g})$*, namely*

$$\Delta_{\mathbb{R}\widetilde{G}} : g \mapsto g \otimes g, \quad \Delta_{\mathfrak{g}} : X_e \mapsto X_e \otimes 1 + 1 \otimes X_e \quad \text{for all } X_e \in \mathfrak{g},$$

while the algebra structure is given by

$$(g \otimes X)(\bar{g} \otimes Y) = (g\bar{g} \otimes (\bar{g}^{-1}X)Y), \tag{7.7}$$

where $\bar{g}^{-1}X := \mathrm{ev}_{\bar{g}^{-1}} * X * \mathrm{ev}_{\bar{g}}$.

Proof. For the coalgebra structure it is enough to check on the generators:

$$\begin{aligned}\Delta(g \otimes X) &= \Delta(g)\Delta(X) \\ &= (g \otimes g)(1 \otimes X + X \otimes 1) \\ &= (g \otimes 1) \otimes (g \otimes X) + (g \otimes X) \otimes (g \otimes 1).\end{aligned}$$

Notice that

$$\psi^{-1}(\Delta(g \otimes X)) = (\mathrm{ev}_g * 1) \otimes (\mathrm{ev}_g * X) + (\mathrm{ev}_g * X) \otimes (\mathrm{ev}_g * 1).$$

On the other hand,

$$\Delta(\psi^{-1}(g \otimes X)) = \Delta(X_g),$$

where

$$\langle \Delta(X_g), f \otimes h \rangle = \langle X_g, f \cdot h \rangle = X_g(f)h + f X_g(h).$$

As for the algebra structure, we have

$$\begin{aligned}(g \otimes X)(\bar{g} \otimes Y) &:= \psi \circ [\psi^{-1}(g \otimes X)\psi^{-1}(\bar{g} \otimes Y)] \\ &= \psi(X_g * Y_{\bar{g}}) \\ &= \psi(\mathrm{ev}_g * X * \mathrm{ev}_{\bar{g}} * Y) \\ &= \psi(\mathrm{ev}_{g\bar{g}} * \mathrm{ev}_{\bar{g}^{-1}} * X * \mathrm{ev}_{\bar{g}} * Y) \\ &= \psi(\mathrm{ev}_{g\bar{g}} * ((\bar{g}^{-1}X) * Y)),\end{aligned}$$

where $\bar{g}^{-1}X := \mathrm{ev}_{\bar{g}^{-1}} * X * \mathrm{ev}_{\bar{g}}$. Therefore we can rewrite our expression as

$$(g \otimes X)(\bar{g} \otimes Y) = (g\bar{g} \otimes (\bar{g}^{-1}X)Y).$$ □

Notice that we have defined a morphism that to each $g \in |G|$ associates the super linear morphism $X \to gX$. This is of fundamental importance and in the ordinary case reduces to the adjoint representation of $\widetilde{G}$ on $\mathfrak{U}(\mathfrak{g})$. We are thus led to the following definition.

Definition 7.3.8. Let G be a super Lie group and $\mathfrak{g}$ the corresponding super Lie algebra. The morphism

$$\mathrm{Ad}\colon \widetilde{G} \to \mathrm{Aut}(\mathfrak{g}), \quad \mathrm{Ad}(g)X := (\mathrm{ev}_g \otimes X \otimes \mathrm{ev}_{g^{-1}})(\mathbb{1} \otimes \mu^*)\mu^*,$$

with $g \in |G|$ and $X \in \mathfrak{g}$, is called the *adjoint representation* of $\widetilde{G}$ on $\mathfrak{g}$. As usual this representation induces a representation of $\widetilde{G}$ on $\mathfrak{U}(\mathfrak{g})$ that is still called adjoint representation.

We now want to see an equivalent formulation of the adjoint representation using the formalism of the functor of points.

Proposition 7.3.9. *For each $g \in |G|$ let us define a morphism $c_g\colon G \to G$, $c_g(x) = gxg^{-1}$ (recall that any topological point of G can be viewed naturally as a T-point of G for all T). Then* $\mathrm{Ad}(g) = (\mathrm{d}c_g)_e$.

Proof. The proof is an instructive application of Yoneda's lemma. By definition we have $(\mathrm{d}c_g)_e X = X \circ c_g^*$, $X \in \mathfrak{g}$. If $x \in G(T) = \mathrm{Hom}(\mathcal{O}(G), \mathcal{O}(T))$, we have

$g \cdot x \cdot g^{-1} = (g \otimes x \otimes g^{-1})(\mu^* \otimes 1)\mu^*$. By Yoneda's lemma, c_g^* corresponds to the image of the identity morphism $1\colon \mathcal{O}(G) \to \mathcal{O}(G)$, hence

$$c_g^*(f) = (g \otimes 1 \otimes g^{-1})(\mu^* \otimes 1)\mu^*(f),$$

where $g\colon \mathcal{O}(G) \to \mathcal{O}(G)$ has to be understood as the evaluation at $g \in |G|$. Thus we have proved our claim (notice that by coassociativity $(\mu^* \otimes 1)\mu^* = (1 \otimes \mu^*)\mu^*$). □

Proposition 7.3.10. *Let the notation be as above. Then* $\mathrm{d\,Ad} = \mathrm{ad}|_{\mathfrak{g}_0}$, *in other words,*

$$\mathrm{d\,Ad}\colon \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}), \quad X \mapsto \{Y \mapsto [X, Y]\}.$$

Proof. This is a direct check. □

Since formula (7.7) resembles the one for semi-direct products, it is customary in the literature to denote $\mathbb{R}\widetilde{G} \otimes \mathfrak{U}(\mathfrak{g})$, endowed with the Hopf algebra structure just described, by

$$\mathbb{R}\widetilde{G} \rtimes \mathfrak{U}(\mathfrak{g}). \tag{7.8}$$

This is also called the *smash product* of $\mathbb{R}\widetilde{G}$ with $\mathfrak{U}(\mathfrak{g})$, but we shall not define smash products in general, referring the interested reader to [72] for more details.

So far we have associated with each super Lie group G the super Hopf algebra $\mathcal{O}(G)^\circ$ of finite support distributions over G, and we have seen that it can be written as

$$\mathcal{O}(G)^\circ \simeq \mathbb{R}\widetilde{G} \rtimes \mathfrak{U}(\mathfrak{g}).$$

The assignment $G \to \mathcal{O}(G)^\circ$ is a faithful embedding from the category of supermanifolds to the category of Hopf superalgebras, however, this embedding is not full. In order to have this, we need to give more structure to $\mathcal{O}(G)^\circ$, which is why we introduce the notion of *super Harish-Chandra pairs* in the next section.

7.4 Super Harish-Chandra pairs

Super Harish-Chandra pairs (SHCP for short) give an equivalent way to approach the theory of Lie supergroups. A SHCP essentially consists of a pair $(G_0, \mathfrak{g})$, where G_0 is an ordinary Lie group and $\mathfrak{g}$ a Lie superalgebra such that $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, together with some natural compatibility conditions. The name comes from an analogy with the theory of Harish-Chandra pairs, that is, the pairs consisting of a compact Lie group K and a Lie algebra $\mathfrak{g}$, with a Cartan involution corresponding to the compact form $\mathfrak{k} = \mathrm{Lie}(K)$. Harish-Chandra pairs are very important in the theory of representation of Lie groups and we shall see in the next chapter that SHCP provide an effective method to study the representations of Lie supergroups.

Definition 7.4.1. Suppose that $(G_0, \mathfrak{g})$ are respectively a Lie group and a super Lie algebra such that

(1) $\mathfrak{g}_0 \simeq \mathrm{Lie}(G_0)$,

(2) G_0 acts on $\mathfrak{g}$ via a representation σ such that $\sigma(G_0)|_{\mathfrak{g}_0} = \mathrm{Ad}$ and the differential of σ acts on $\mathfrak{g}$ as the adjoint representation, that is,

$$\mathrm{d}\sigma(X)Y = [X, Y].$$

Then the pair $(G_0, \mathfrak{g})$ is called a *a super Harish-Chandra pair* (SHCP). We shall write $(G_0, \mathfrak{g}, \sigma)$ when we want to stress the action σ.

A morphism of SHCPs is simply a pair of morphisms $\psi = (\psi_0, \rho^\psi)$ preserving the SHCP structure.

Definition 7.4.2. Let $(G_0, \mathfrak{g}, \sigma)$ and $(H_0, \mathfrak{h}, \tau)$ be SHCP. A *morphism* between them is a pair (ψ_0, ρ^ψ) such that

(1) $\psi_0 \colon G_0 \to H_0$ is a Lie groups homomorphism,

(2) $\rho^\psi \colon \mathfrak{g} \to \mathfrak{h}$ is a super Lie algebra homomorphism,

(3) ψ_0 and ρ^ψ are compatible in the sense that

$$\rho^\psi|_{\mathfrak{g}_0} \simeq (\mathrm{d}\psi_0)_e, \quad \tau(\psi_0(g)) \circ \rho^\psi = \rho^\psi \circ \sigma(g).$$

Example 7.4.3. If G is a super Lie group, the pair $(\widetilde{G}, \mathfrak{g})$ given by the reduced Lie group of G and the super Lie algebra $\mathfrak{g}$ is a super Harish-Chandra pair with respect to the adjoint action of $\widetilde{G}$ on $\mathfrak{g}$ as defined in Definition 7.3.8. Moreover, given a morphism $\phi \colon G \to H$ of super Lie groups, ϕ determines the morphism of the corresponding super Harish-Chandra pairs

$$(|\phi|, (\mathrm{d}\phi)_e).$$

In the next observation we relate the SHCP with the Kostant construction described in the previous section.

Observation 7.4.4. Given an SHCP pair, we can construct the super Hopf algebra given by the semidirect product $\mathbb{R}G_0 \rtimes \mathfrak{U}(\mathfrak{g})$ in which G_0 is endowed with a Lie group structure and the product is given by

$$(g_1, X_e) * (g_2, Y_e) := (g_1 g_2, (\sigma(g_2)^{-1} X_e) Y_e).$$

This is a straightforward generalization of (7.8).

Such objects were introduced by Kostant in [49] under the name of *Lie–Hopf superalgebras*. The category of Lie–Hopf superalgebras is clearly isomorphic to the category of SHCP, and given a super Lie group G we can associate to it the Lie–Hopf superalgebra $\mathcal{O}(G)^\circ \simeq \mathbb{R}\widetilde{G} \rtimes \mathfrak{U}(\mathfrak{g})$ in which $\widetilde{G}$ is the associated classical Lie group.

We summarize our previous considerations by saying that we have defined a functor

$$\mathcal{H}\colon \textbf{SGrp} \to (\text{shcps}), \quad G \mapsto (\widetilde{G}, \mathfrak{g}, \mathrm{Ad}),$$

from the category of super Lie groups to the category of super Harish-Chandra pairs. The most important result of this chapter is the following:

Theorem 7.4.5. *The category of super Lie groups is equivalent to the category of super Harish-Chandra pairs.*

Roughly speaking this theorem says that each problem in the category of super Lie groups can be reformulated as an equivalent problem in the language of SHCP. Before embarking on the proof, let us outline the path that we shall follow. We show the following.

(i) Given a SHCP $(G_0, \mathfrak{g})$ there exists a super Lie group G whose associated SHCP is isomorphic to $(G_0, \mathfrak{g})$.

(ii) Given a morphism of SHCP $(\psi_0, \rho_\psi)\colon (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$ there exists a unique morphism ψ of the corresponding super Lie groups from which (ψ_0, ρ_ψ) arises.

iii) Due to points (i) and (ii) we have a functor

$$\mathcal{K}\colon (\text{shcps}) \to \textbf{SGrp}.$$

In order to prove the theorem we have to show that $\mathcal{K} \circ \mathcal{H} \simeq 1_{\textbf{SGrp}}$ and $\mathcal{H} \circ \mathcal{K} \simeq 1_{(\text{shcps})}$. This means that, for each $G \in \textbf{SGrp}$ and $(G_0, \mathfrak{g}) \in (\text{shcps})$, $(\mathcal{K} \circ \mathcal{H})(G) \simeq G$ and $(\mathcal{H} \circ \mathcal{K})((G_0, \mathfrak{g})) \simeq (G_0, \mathfrak{g})$, and moreover the diagrams

$$\begin{array}{ccc} (\mathcal{K} \circ \mathcal{H})(G) & \xrightarrow{\sim} & G \\ {\scriptstyle (\mathcal{K}\circ\mathcal{H})(\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\ (\mathcal{K} \circ \mathcal{H})(H) & \xrightarrow{\sim} & H, \end{array} \qquad \begin{array}{ccc} (\mathcal{H} \circ \mathcal{K})((G_0, \mathfrak{g})) & \xrightarrow{\sim} & (G_0, \mathfrak{g}) \\ {\scriptstyle (\mathcal{H}\circ\mathcal{K})(\chi)}\downarrow & & \downarrow{\scriptstyle \chi} \\ (\mathcal{H} \circ \mathcal{K})((H_0, \mathfrak{h})) & \xrightarrow{\sim} & (H_0, \mathfrak{h}) \end{array} \tag{7.9}$$

commute for each $\phi\colon G \to H$ and $\chi\colon (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$.

We start with the reconstruction of a super Lie group from an SHCP. Suppose that an SHCP $(G_0, \mathfrak{g}, \sigma)$ is given and notice that:

(1) $\mathfrak{U}(\mathfrak{g})$ is naturally a left $\mathfrak{U}(\mathfrak{g}_0)$-module.

(2) For each open set $U \subseteq G_0$, $C^\infty_{G_0}(U)$ is a left $\mathfrak{U}(\mathfrak{g}_0)$-module. In fact (see for example, [75]) each $X \in \mathfrak{g}_0$ acts from the left on smooth functions on G_0 as the left-invariant differential operator[1]

$$(\widetilde{D}^L_X f)(g) := \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{tX})\Big|_{t=0}.$$

[1] Notice that, as already remarked, here and in the following we do not mention explicitly the isomorphism $\mathrm{Lie}(G_0) \simeq \mathfrak{g}_0$ appearing in the definition of an SHCP.

Hence, for each open subset $U \subseteq G_0$, we can define the assignment

$$U \mapsto \mathcal{O}_G(U) := \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty_{G_0}(U)),$$

where the right-hand side is the subset of $\underline{\mathrm{Hom}}(\mathfrak{U}(\mathfrak{g}), C^\infty_{G_0}(U))$ consisting of $\mathfrak{U}(\mathfrak{g}_0)$-linear morphisms. (Notice that, for the moment, G is just a letter and we have not defined any supergroup structure on G_0.)

Remark 7.4.6. If $\mathfrak{g} = \mathfrak{g}_0$ we have

$$\underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty_{G_0}(U)) \cong C^\infty_{G_0}(U).$$

In fact a $\mathfrak{U}(\mathfrak{g}_0)$-linear map is uniquely determined by the image of $1 \in \mathfrak{U}(\mathfrak{g})$ and consequently we can uniquely associate to any morphism an element of $C^\infty_{G_0}(U)$.

$\mathcal{O}_G(U)$ has a natural structure of unital, commutative superalgebra. The multiplication $\mathcal{O}_G(U) \otimes \mathcal{O}_G(U) \to \mathcal{O}_G(U)$ is defined by

$$f_1 \cdot f_2 := m_{C^\infty(G_0)} \circ (f_1 \otimes f_2) \circ \Delta_{\mathfrak{U}(\mathfrak{g})} \tag{7.10}$$

and the unit can be identified (with a mild abuse of notation) with the counit ϵ of $\mathfrak{U}(\mathfrak{g})$. We leave to the reader the simple check that $f_1 \cdot f_2 \in \mathcal{O}_G(U)$.

If U and V are open subsets of G_0 such that $U \subseteq V$, we define the superalgebra morphism

$$\rho_{V,U} \colon \mathcal{O}_G(V) \to \mathcal{O}_G(U), \quad f \mapsto \tilde{\rho}_{V,U} \circ f, \tag{7.11}$$

where $\tilde{\rho}_{V,U} \colon C^\infty(V) \to C^\infty(U)$ is the usual restriction map. We will often abbreviate $\rho_{V,U}(f)$ with $f|_U$.

Lemma 7.4.7. *$\mathcal{O}_G$, together with the restriction maps (7.11), is a sheaf of superalgebras.*

Proof. The fact that $\mathcal{O}_G$ is a presheaf is a routine check. As for the glueing property, let $\{U_\alpha\}$ be open sets covering a fixed open set U and f_α elements in $\mathcal{O}_G(U_\alpha)$ such that $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for each α, β. We want to define $f \in \mathcal{O}_G(U)$. For each $X \in \mathfrak{U}(\mathfrak{g})$, the $f_\alpha(X) \in C^\infty(U_\alpha)$ glue together to give $g_X \in C^\infty(U)$. Define $f(X) = g_X$. Then $\mathfrak{U}(\mathfrak{g}_0)$-linearity is immediate. □

Lemma 7.4.8. *The antisymmetrizer*

$$\gamma \colon \Lambda(\mathfrak{g}_1) \to \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_p \mapsto \frac{1}{p!} \sum_{\tau \in S_p} (-1)^{|\tau|} X_{\tau(1)} \cdots X_{\tau(p)},$$

is a super coalgebra morphism, where S_p denotes the group of permutations of p elements.

The map

$$\hat{\gamma} \colon \mathfrak{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1) \to \mathfrak{U}(\mathfrak{g}), \quad X \otimes Y \mapsto X \cdot \gamma(Y),$$

is an isomorphism of super left $\mathfrak{U}(\mathfrak{g}_0)$-modules.

Proof. Both assertions are consequences of the PBW theorem and we leave their proofs to the reader. □

Proposition 7.4.9. *$(G_0, \mathcal{O}_G)$ is a supermanifold that is globally split, i.e., for each open subset $U \subseteq G_0$ there is an isomorphism of superalgebras*

$$\mathcal{O}_G(U) \simeq \underline{\mathrm{Hom}}(\bigwedge(\mathfrak{g}_1), C^\infty(U)) \simeq C^\infty(U) \otimes \bigwedge(\mathfrak{g}_1)^*.$$

Hence $\mathcal{O}_G$ carries a natural $\mathbb{Z}$-gradation.

Proof. In view of Lemma 7.4.7, it only remains to prove the local triviality of the sheaf. For this purpose we define the map

$$\phi_U : \mathcal{O}_G(U) \to \underline{\mathrm{Hom}}(\bigwedge(\mathfrak{g}_1), C^\infty(U)), \quad f \to f \circ \gamma.$$

Since γ is a supercoalgebra morphism, ϕ_U is a superalgebra morphism. In fact,

$$\phi_U(f_1 \cdot f_2) = m \circ f_1 \otimes f_2 \circ \Delta_{\mathfrak{U}(\mathfrak{g})} \circ \gamma = m \circ f_1 \otimes f_2 \circ (\gamma \otimes \gamma)\Delta_{\mathfrak{U}(\mathfrak{g})} = \phi_U(f_1)\phi_U(f_2).$$

That ϕ_U is a superalgebra isomorphism follows at once from $\mathfrak{U}(\mathfrak{g}_0)$-linearity. □

So far we have used only the fact that $C^\infty_{G_0}(U)$ is a left $\mathfrak{U}(\mathfrak{g}_0)$-module.

The next proposition uses more heavily the structure of G_0 and the representation σ. It exhibits explicitly the structure of a super Lie group in terms of the corresponding SHCP.

Proposition 7.4.10. *$(G_0, \mathcal{O}_G)$ is a super Lie group with respect to the morphisms multiplication $\mu\colon G \times G \to G$, inverse $i\colon G \to G$ and unit $e\colon k \to G$, which are defined in terms of their pullbacks μ^*, i^*, e^* as*

$$\begin{aligned} [\mu^*(f)(X,Y)](g,h) &= [f((h^{-1} \cdot X)Y)](gh), \\ [i^*(f)(X)](g^{-1}) &= [f(g^{-1} \cdot \bar{X})](g), \\ e^*(f) &= [f(1)](e), \end{aligned} \tag{7.12}$$

where $X, Y \in \mathfrak{U}(\mathfrak{g})$, $g, h \in G_0$, $f \in \mathcal{O}_G(U)$, e is the unit of G_0, $g \cdot X := \sigma(g)X$ and $\bar{X}$ denotes the antipode of X in $\mathfrak{U}(\mathfrak{g})$.

Proof. The proposition will follow if we prove the following statements:

i) μ^* is well defined, a superalgebra morphism and associative;

ii) e^* is well defined, a superalgebra morphism and a unit for μ^*;

iii) i^* is well defined, a superalgebra morphism and an antipode for μ^*.

The verification of these assertions consists in a quantity of long and tedious calculations. We hence select the most significant of them, leaving the others to the reader. Let us begin with the various steps.

i) *Comultiplication*. In order to show that μ^* is well defined, we have to prove that if $f \in \mathcal{O}_G(U)$, $\mu^*(f)$ belongs to $\underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g}), C^\infty(W))$, for a suitable W open in $G_0 \times G_0$. Due to PBW theorem, it is enough to prove $\mathfrak{g}_0$-linearity. Let $Z \in \mathfrak{g}_0$. Then

$$\begin{aligned} \mu^*(f)(ZX, Y)(g,h) &= f(h^{-1}(ZX)Y)(gh) \\ &= f((h^{-1} \cdot Z)(h^{-1} \cdot X)Y)(gh) \\ &= \widetilde{D}^L_{h^{-1} \cdot Z}[f((h^{-1} \cdot X)Y)](gh). \end{aligned}$$

On the other hand,

$$\begin{aligned} [(\widetilde{D}^L_Z \otimes \mathbb{1})(\mu^*(f)(X,Y))](g,h) &= \tfrac{d}{dt}\big|_{t=0} f((h^{-1}X)Y)(ge^{tZ}h) \\ &= \tfrac{d}{dt}\big|_{t=0} f((h^{-1}X)Y)(ghe^{t(h^{-1}Z)}) \\ &= \widetilde{D}^L_{h^{-1}Z}[f((h^{-1} \cdot X)Y)](gh). \end{aligned}$$

Similarly for the left entry, one finds

$$\begin{aligned} \mu^*(f)(X, ZY)(g,h) &= f((h^{-1}X)ZY)(gh) \\ &= f(Z(h^{-1}X)Y + [h^{-1}X, Z]Y)(gh) \\ &= \widetilde{D}^L_Z(f((h^{-1}X)Y))(gh) + f([h^{-1}X, Z]Y)(gh) \end{aligned}$$

and

$$\begin{aligned} [(\mathbb{1} \otimes \widetilde{D}^L_Z)(\mu^*(f)(X,Y))](g,h) &= \tfrac{d}{dt}\big|_{|t=0} \mu^*(f)(X,Y)(g, he^{tZ}) \\ &= \tfrac{d}{dt}\big|_{t=0} f(((he^{tZ})^{-1}X)Y)(ghe^{tZ}) \\ &= [\widetilde{D}^L_Z f((h^{-1}X)Y)](gh) + f([(h^{-1}X), Z]Y)(gh) \end{aligned}$$

where

$$\tfrac{d}{dt}\big|_{t=0} \sigma((he^{tZ})^{-1})X = \tfrac{d}{dt}\big|_{t=0} \sigma(e^{-tZ})\sigma(h^{-1})X = \mathrm{d}\sigma(-Z)h^{-1}X = [h^{-1}X, Z].$$

We leave it to the reader to check that μ^* is a superalgebra morphism.

We now show that μ^* is associative. Indeed,

$$\begin{aligned} ((\mu^* \otimes \mathbb{1}) \circ \mu^*)(f)(X,Y,Z)(g,h,k) &= \mu^*(f)((h^{-1}X)Y, Z)(gh,k) \\ &= f[k^{-1}((h^{-1}X)Y)Z](ghk) \\ &= f((k^{-1}h^{-1}X)(k^{-1}Y)Z)(ghk) \end{aligned}$$

and

$$\begin{aligned} ((\mathbb{1} \otimes \mu^*) \circ \mu^*)(f)(X,Y,Z)(g,h,k) &= \mu^*(f)(X, (k^{-1}Y)Z)(g, hk) \\ &= f((k^{-1}h^{-1}X)(k^{-1}Y)Z)(ghk). \end{aligned}$$

ii) *Counit*. We need to check the counit property, that is,

$$(e^* \otimes \mathbb{1})\mu^*(f) = (\mathbb{1} \otimes e^*)\mu^*(f) = f.$$

This is completely straightforward and is left to the reader.

iii) *Antipode*. First we show that it is well defined. Again due to the PBW theorem we can take $Z \in \mathfrak{h}_0$. Then

$$\begin{aligned} i^*(f)(ZX)(g^{-1}) &= f(g^{-1}(\overline{ZX}))(g) \\ &= f(g^{-1}(\bar{Z}\bar{X} + [\bar{X}, \bar{Z}]))(g) \\ &= f((g^{-1}\bar{Z})(g^{-1}\bar{X}) + g^{-1}[X, Z])(g) \\ &= [\tilde{D}^L_{g^{-1}\bar{Z}} f(g^{-1}\bar{X})](g) + f(g^{-1}[X, Z])(g) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} i^*(f)(X)(g^{-1}e^{tZ}) &= \frac{d}{dt}\Big|_{|t=0} i^*(f)(X)(e^{t\bar{Z}}g)^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} f((g^{-1}e^{tZ}) \cdot \bar{X})(e^{t\bar{Z}}g) \\ &= \tilde{D}^L_{g^{-1}\bar{Z}} f(g^{-1}\bar{X})(g) + f(g^{-1}[X, Z])(g). \end{aligned}$$

It is a superalgebra morphism since

$$\begin{aligned} i^*(f_1 \cdot f_2)(X)(g^{-1}) &= (f_1 \cdot f_2)(g^{-1}\bar{X})(g) \\ &= [(f_1 \otimes f_2)\Delta(g^{-1}\bar{X})](g, g) \end{aligned}$$

and

$$\begin{aligned} (i^*(f_1) \cdot i^*(f_2))(X)(g^{-1}) &= [(i^*(f_1) \otimes i^*(f_2))\Delta X](g^{-1}, g^{-1}) \\ &= [(f_1 \otimes f_2)(\sigma(g^{-1}) \otimes \sigma(g^{-1})) \\ &\quad \cdot (S_{\mathfrak{U}(\mathfrak{g}_0)} \otimes S_{\mathfrak{U}(\mathfrak{g}_0)})\Delta(X)](g, g) \\ &= [(f_1 \otimes f_2)\Delta(g^{-1}\bar{X})](g, g). \end{aligned}$$

We now show that i^* is a right antipode.

$$\begin{aligned} [(m \circ (\mathbb{1} \otimes i^*) \circ \mu^*)(f)(X)](g) &= [\mu^*(f)(\mathbb{1} \otimes \sigma(g))(\mathbb{1} \otimes S_{\mathfrak{U}(\mathfrak{g}_0)})\Delta(X)](g, g^{-1}) \\ &= [f(m(\sigma(g) \otimes \sigma(g))(\mathbb{1} \otimes S_{\mathfrak{U}(\mathfrak{g}_0)})\Delta(X))](e) \\ &= [f(\sigma(g)m((\mathbb{1} \otimes S_{\mathfrak{U}(\mathfrak{g}_0)})\Delta(X))](e) \\ &= [f\sigma(g)\epsilon(X)](e). \end{aligned}$$

A similar computation shows that i^* is also a left antipode. □

Let us now collect a glossary of some frequently used operations in our realization, completing those given in equations (7.10) and (7.12). Notice that since $(-1)^{|X|(|f|+|Y|)} f(YX) = (-1)^{|X|} f(YX)$, it is possible to slightly simplify the form of some expressions and this is a consequence of the trivial fact that $f(Z) = 0$ unless $|f| = |Z|$.

Lemma 7.4.11. *Let the notation be as above, and let* $f \in \mathcal{O}_G(U)$*,* $g, h \in |G|$*,* $X, Y \in \mathcal{U}(\mathfrak{g})$*. Then we have the following operations:*

(1) *Evaluation map*: $|f| = f(1)$, $\mathrm{ev}_g(f) = f(1)(g) = |f|(g)$.
(2) *Left translation*: $[\ell_h^*(f)](X) = |\ell_h^*|(f(X))$.
(3) *Right translation*: $\big[r_h^*(f)\big](X) = |r_h^*|(f(h^{-1} \cdot X))$.
(4) *Left invariant vector fields*: $(D_X^L f)(Y) = (-1)^{|X|} f(YX)$.
(5) *Right invariant vector fields*: $[(D_X^R f)(Y)](g) = (-1)^{|X||f|} f((g^{-1} \cdot X)Y)(g)$.

Proof. (1) is clear from the definitions.

As for (2) we have

$$\begin{aligned}
\big[\ell_h^*(f)\big](X)(g) &= [(\mathrm{ev}_h \otimes \mathbb{1}) \circ \mu^*(f)](X)(g) \\
&= [(\mathrm{ev}_h \otimes \mathbb{1}) \sum (f^{(1)} \otimes f^{(2)})](X)(g) \\
&= \sum f^{(1)}(1)(h) f^{(2)}(X)(g) = \mu^*(f)(1, X)(h, g) \\
&= f(X)(hg) = |\ell_h^*|(f(X)).
\end{aligned}$$

(3) is very similar; in fact we have

$$\begin{aligned}
\big[r_h^*(f)\big](X) &= [(\mathbb{1} \otimes \mathrm{ev}_h) \circ \mu^*(f)](X)(g) \\
&= [(\mathbb{1} \otimes \mathrm{ev}_h) \sum (f^{(1)} \otimes f^{(2)})](X)(g) \\
&= \sum f^{(1)}(X)(g) f^{(2)}(1)(h) = \mu^*(f)(X, 1)(g, h) \\
&= f(h^{-1}X)(gh) = |r_h^*|(f(h^{-1} \cdot X)).
\end{aligned}$$

(4) We first check that the vector field $(D_X^L f)(Y) := (-1)^{|X|} f(YX)$ is left-invariant, that is, $(\mathbb{1} \otimes D_X^L)\mu^* = \mu^* D_X^L$:

$$\begin{aligned}
[(\mathbb{1} \otimes D_X^L)\mu^*(f)](Y, Z)(g, h) &= (-1)^{|X|} \mu^*(f)(Y, ZX)(g, h) \\
&= (-1)^{|X|} f(h^{-1}YZX)(gh)
\end{aligned}$$

and

$$\begin{aligned}
[\mu^* D_X^L f](Y, Z)(g, h) &= D_X^L(f)(h^{-1}YZ)(gh) \\
&= (-1)^{|X|} f(h^{-1}YZX)(gh).
\end{aligned}$$

Hence we have a well-defined linear map from the Lie superalgebra $\mathfrak{g}$ and the left-invariant vector fields $X \mapsto D_X^L$. In order to show that it is an isomorphism it is enough to prove injectivity for dimension considerations. If $X \neq Y$ are two elements in $\mathfrak{g}$, with $D_X^L(f) = D_Y^L(f)$ for all f, then $f(X) = f(Y)$ for all X and Y, reaching a contradiction.

(5) For right-invariant vector fields the arguments are very similar and the proofs are left to the reader. $\square$

Notice that we have introduced the notation D_X^L, D_X^R for the actions of left and right-invariant vector fields, which is different from the one used in the previous sections.

With this approach it is very natural to recover a super Lie group morphism from an SHCP one. Suppose that (ψ_0, ρ_ψ) is a morphism from $(G_0, \mathfrak{g})$ to $(H_0, \mathfrak{h})$, and that $f \in \mathcal{O}_H(U)$. Define $\psi^*(f)$ via the diagram

$$\begin{array}{ccc} \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\rho_\psi} & \mathfrak{U}(\mathfrak{h}) \\ {\scriptstyle \psi^*(f)} \downarrow & & \downarrow {\scriptstyle f} \\ C^\infty_{G_0}(\psi_0^{-1}(U)) & \xleftarrow[\psi_0^*]{} & C^\infty_{H_0}(U). \end{array}$$

It is not difficult to prove that this defines a super Lie group morphism with associated SHCP morphism (ψ_0, ρ_ψ). Indeed we have the following proposition.

Proposition 7.4.12. *The map*

$$\begin{aligned} \psi^*\colon\ \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h}_0)}(\mathfrak{U}(\mathfrak{h}), C^\infty(H_0)) &\to \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty(G_0)), \\ f &\mapsto \psi^*(f) := \psi_0^* \circ f \circ \rho_\psi, \end{aligned}$$

defines a morphism of super Lie groups whose reduced morphism is ψ_0 and whose differential at the identity is ρ_ψ.

Proof. It is immediate that ψ^* is well defined. We check that the above defined map is a map of superalgebras and that it is a super Lie group morphism. It is a superalgebra morphism since, by Remark 7.4.6,

$$\begin{aligned} \psi^*(f_1 \cdot f_2)(X)(g) &:= [m_{C^\infty(G_0)}(f_1 \otimes f_2)\Delta\rho_\psi(X)](\psi_0(g)) \\ &= [(f_1 \otimes f_2)(\rho_\psi \otimes \rho_\psi)\Delta(X)](\psi_0(g), \psi_0(g)) \\ &= [\psi^*(f_1) \cdot \psi^*(f_2)](X)(g). \end{aligned}$$

It is a super Lie group morphism; indeed compare

$$\begin{aligned} \mu^*(\psi^*(f))(X, Y)(g, h) &= [\psi^*(f)((h^{-1}X)Y)](gh) \\ &= [f(\rho_\psi((\psi_0(h)^{-1}X))\rho_\psi(Y))](\psi_0(gh)) \end{aligned}$$

with

$$\begin{aligned} [(\psi^* \otimes \psi^*)\mu^*(f)](X, Y)(g, h) &= [\mu^*(f)(\rho_\psi(X), \rho_\psi(Y))](\psi_0(g), \psi_0(h)) \\ &= [f(\rho_\psi(\psi_0(h^{-1})X)\rho_\psi(Y))](\psi_0(gh)) \end{aligned}$$

and taking into account the obvious properties of ψ_0 and ρ_ψ.

It is also well behaved with respect to the inverse:

$$\begin{aligned} \psi^*(i^*(f))(X)(g^{-1}) &= i^*(f)(\rho_\psi(X))(\psi_0(g)) \\ &= f(\psi_0(g)^{-1}\overline{\rho_\psi(X)})(\psi_0(g)) \end{aligned}$$

and

$$
\begin{aligned}
i^*(\psi^*(f))(X)(g^{-1}) &= f(\rho_\psi(g^{-1}\overline{X}))(\psi_0(g)) \\
&= f(\psi_0(g)^{-1}\rho_\psi(\overline{X}))(\psi_0(g)).
\end{aligned}
$$

We now compute the differential $(\mathrm{d}\psi)_e$ at the identity and show that $(\mathrm{d}\psi)_e = \rho_\psi$. Let χ be an element of the Lie superalgebra $\mathfrak{h} = T_e H$. Then, using various properties of the previous lemma, we obtain

$$
\begin{aligned}
(\mathrm{d}\psi)_e(\chi)(f) &= [\chi \circ \psi_0^*](f) \\
&= \chi \circ \psi_0^* \circ f \circ \rho_\psi \\
&= \mathrm{ev}_e \circ D^L_\chi \circ \psi_0^* \circ f \circ \rho_\psi \\
&= (D^L_\chi \circ \psi_0^* \circ f \circ \rho_\psi)(1)(e) \\
&= (-1)^{|\chi|}(\psi_0^* \circ f \circ \rho_\psi)(\chi)(e) \\
&= (-1)^{|\chi|}[f(\rho_\psi(\chi))](\psi_0(e)) \\
&= (-1)^{|\chi|}[f(\rho_\psi(\chi))](e) \\
&= (-1)^{|\chi|+|\rho_\psi(\chi)|}[D^L_{\rho_\psi(\chi)} f](1)(e) \\
&= \rho_\psi(\chi)(f).
\end{aligned}
$$

□

We can finally end the proof of Theorem 7.4.5, proving the following proposition.

Proposition 7.4.13. *Let the notation be as above. Define the functors*

$$
\begin{aligned}
&\mathcal{H}\colon \textbf{SGrp} \to (\text{shcps}), \quad G \to (\widetilde{G}, \mathfrak{g}, \mathrm{Ad}), \quad \phi \to (|\phi|, (\mathrm{d}\phi)_e), \\
&\mathcal{K}\colon (\text{shcps}) \to \textbf{SGrp}, \quad (G_0, \mathfrak{g}, \sigma) \to \overline{G} = (G_0, \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty_{G_0})), \\
&\psi \to f \mapsto \psi_0^* \cdot f \cdot \rho_\psi,
\end{aligned}
$$

where G and $(G_0, \mathfrak{g}, \sigma)$ are objects and ϕ, ψ are morphisms of the corresponding categories (for the notation relative to ψ see Proposition 7.4.12).

Then $\mathcal{H}$ and $\mathcal{K}$ define an equivalence between the categories of super Lie groups and super Harish-Chandra pairs.

Proof. We first check that $(\mathcal{H} \circ \mathcal{K})(G_0, \mathfrak{g}) \simeq (G_0, \mathfrak{g})$. Clearly $G_0 = |\overline{G}|$, moreover the two equalities

$$
\begin{aligned}
[D^L_X D^L_Y - (-1)^{|X||Y|} D^L_Y D^L_X] &= D^L_{[X,Y]}, \\
r_g^* \, D^L_X \, r^*_{g^{-1}} &= D^L_{g \cdot X}
\end{aligned}
$$

for each $X, Y \in \mathfrak{g}$ and $g \in G_0$ tell us that the Lie superalgebras $\mathrm{Lie}(G)$ and $\mathfrak{g}$ are the same and so is the action of G_0 on $\mathfrak{g}$.

We now turn to check that $\overline{G} = (\mathcal{K} \circ \mathcal{H})(G) \simeq G$. In order to do this we shall build a morphism $\eta\colon \overline{G} \to G$ and prove that it is an SLG isomorphism.

- Let η be defined by the pullback

$$\eta^* : \mathcal{O}(G) \to \mathcal{O}(\bar{G}) = \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty(G_0)),$$
$$s \mapsto (\bar{s}\colon X \to (-1)^{|X|}|(D_X^L s)|).$$

Here D_X^L denotes the left-invariant differential operator on G associated with $X \in \mathfrak{U}(\mathfrak{g})$, that is, $D_X^L = (1 \otimes X)\mu^*$.

- The definition is well posed. Indeed,

$$\bar{s}\colon X_0 X \mapsto (-1)^{|X|}|(D_{X_0}^L D_X^L s)| = (-1)^{|X|}\widetilde{D}_{X_0}^L|(D_X^L s)|$$

for each $X_0 \in \mathfrak{g}_0$ and $X \in \mathfrak{U}(\mathfrak{g})$, so $\bar{s}$ is actually $\mathfrak{U}(\mathfrak{g}_0)$-linear, and moreover,

$$\eta^*(s)\eta^*(t) = \eta^*(st)$$

for each $s, t \in \mathcal{O}(G)$ and $X \in \mathfrak{U}(\mathfrak{g})$ since

$$\begin{aligned}(\bar{s} \cdot \bar{t})(X) &= m_{C^\infty_{G_0}}(\bar{s} \otimes \bar{t})\Delta(X)\\ &= \sum (-1)^{|t||X_{(1)}|}(-1)^{|X_{(1)}|}(-1)^{|X_{(2)}|}|(D_{X_{(1)}}^L s)||(D_{X_{(2)}}^L t)|\\ &= \sum (-1)^{|t||X_{(1)}|}(-1)^{|s||X_{(2)}|}(-1)^{|X|}|(D_X^L st)|\\ &= \overline{st}(X),\end{aligned}$$

where $\sum X_{(1)} \otimes X_{(2)} = \Delta(X)$.

- η is an SLG morphism, i.e.,

$$\eta \circ \mu_{\bar{G}} = \mu_G \circ (\eta \times \eta).$$

Indeed, for each $s \in \mathcal{O}(G)$, $X, Y \in \mathfrak{U}(\mathfrak{g})$, and $g, h \in G_0$,

$$\begin{aligned}[((\eta^* \otimes \eta^*)\mu_G^*(s))(X, Y)](g, h) &= (-1)^{|X|+|Y|}|(D_X^L \otimes D_Y^L)\mu_G^*(s)|(g, h)\\ &= (-1)^{|X|+|Y|}|D_{h^{-1}.X}^L D_Y^L s|(gh)\\ &= [\eta^*(s)\big((h^{-1}.X)Y\big)](gh)\\ &= [(\mu_{\bar{G}}^*\eta^*(s))(X, Y)](g, h).\end{aligned}$$

- η is an isomorphism. This is due to Corollary 5.1.3 since $|\eta|$ is clearly bijective and, for each $g \in G_0$, the differential $(d\eta)_g$ is bijective as

$$\begin{aligned}[(\mathrm{d}\eta)_g(\bar{D}_X^L|_g)](s) &= \bar{D}_X^L|_g \eta^*(s)\\ &= \mathrm{ev}_g(\bar{D}_X^L \eta^*(s))\\ &= [\bar{D}_X^L \eta^*(s)](1)(g)\\ &= (-1)^{|X|}\eta^*(s)(X)(g)\\ &= |(D_X^L s)|(g)\\ &= D_X^L|_g(s),\end{aligned}$$

where we denote by $\bar{D}_X^L$ a left-invariant differential operator on $\bar{G}$ corresponding to $X \in \mathfrak{U}(\mathfrak{g})$ while D_X^L denotes a left-invariant differential operator on G. Notice also that if $\{X_i\}$ is a basis of $\mathfrak{g}$, then $D_{X_i}^L|_g := \mathrm{ev}_g D_{X_i}^L$, $g \in |G| = |\bar{G}|$, forms a basis of $T_g(\bar{G})$.

It only remains to check the commutative diagrams (7.9). If

$$(\phi_0, \rho_\phi)\colon (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$$

is an SHCP morphism, then (ϕ_0, ρ_ϕ) induces $\bar{\phi}\colon \bar{G} \to \bar{H}$ and we want to check that $|\bar{\phi}| = \phi_0$ and $\mathrm{d}\bar{\phi} = \rho_\phi$. Clearly $|\bar{\phi}| = \phi_0$ and, for each $X \in \mathfrak{h}$ and $f \in \mathcal{O}(\bar{H})$,

$$\begin{aligned}[d\bar{\phi}(X)](f) &= \mathrm{ev}_e D_X^L(\bar{\phi}^*(f)) \\ &= (-1)^{|X|}[\phi_0^* f \rho_\phi(X)](e) \\ &= (D_{\rho_\phi(X)}^L f)(e).\end{aligned}$$

On the other hand, if

$$\phi\colon G \to H$$

is an SLG morphism and $\bar{\phi} = (\mathcal{K} \circ \mathcal{H})(\phi)$, we want to check that $\bar{\phi}^*\eta^* = \eta^*\phi^*$. For each $s \in \mathcal{O}(H)$, $X \in \mathfrak{U}(\mathfrak{g})$, and $g \in |G|$, we have

$$\begin{aligned}[(\bar{\phi}^*\eta^*(s))(X)](g) &= (-1)^{|X|}[|\phi|^*(|D_{(\mathrm{d}\phi)_e(X)}^L s|)](g) \\ &= (-1)^{|X|}|\phi^*(D_{(\mathrm{d}\phi)_e(X)}^L s)|(g) \quad (\text{since } |\phi|^*(|s|) = |\phi^*(s)| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for each } s \in \mathcal{O}(H)) \\ &= (-1)^{|X|}[\phi^*(\mathbb{1} \otimes (\mathrm{d}\phi)_e(X))\mu_H^*(s)](g) \\ &= (-1)^{|X|}[(\mathbb{1} \otimes X)(\phi^* \otimes \phi^*)\mu_H^*(s)](g) \\ &= (-1)^{|X|}[(\mathbb{1} \otimes X)\mu_G^*\phi^*(s)](g) \quad (\text{since } \phi \text{ is an SLG morphism}) \\ &= (-1)^{|X|}[D_X^L\phi^*(s)](g) \\ &= [(\eta^*\phi^*(s))(X)](g).\end{aligned}$$

(Notice that $D_{(\mathrm{d}\phi)_e(X)}^L$ is on H and D_X^L on G.) □

In the next example we show explicitly how to recover the group structure of a super Lie group starting from its associated SHCP.

Example 7.4.14. We consider the super Lie group $G = \mathrm{GL}(1|1)$. In the language of T-points, $\mathrm{GL}(1|1)(T)$ is the set of invertible matrices $\begin{pmatrix} x_1 & \theta_1 \\ \theta_2 & x_2 \end{pmatrix}$ with multiplication

$$\begin{pmatrix} x_1 & \theta_1 \\ \theta_2 & x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 & \xi_1 \\ \xi_2 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 + \theta_1 \xi_2 & x_1 \xi_1 + \theta_1 y_2 \\ \theta_2 y_1 + x_2 \xi_2 & x_2 y_2 + \theta_2 \xi_1 \end{pmatrix}, \qquad (7.13)$$

with $x_i \in \mathcal{O}(T)_0$, $\xi_i \in \mathcal{O}(T)_1$ for a supermanifold T. The corresponding reduced group is $|G| = (\mathbb{R} \setminus \{0\})^2$. A basis for the left-invariant vector fields $\mathfrak{gl}(1|1)$ is easily recognized to be

$$X_1 = x_1 \frac{\partial}{\partial x_1} + \theta_2 \frac{\partial}{\partial \theta_2}, \quad X_2 = x_2 \frac{\partial}{\partial x_2} + \theta_1 \frac{\partial}{\partial \theta_1},$$
$$\Theta_1 = x_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial x_2}, \quad \Theta_2 = x_2 \frac{\partial}{\partial \theta_2} - \theta_1 \frac{\partial}{\partial x_1},$$

with commutation relations (for all $i, j = 1, 2$)

$$[X_i, X_j] = 0, \qquad [\Theta_i, \Theta_i] = 0,$$
$$[X_i, \Theta_j] = (-1)^{i+j}\Theta_j, \quad [\Theta_1, \Theta_2] = -X_1 - X_2.$$

The element $h = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \in |G|$ acts through the adjoint representation on $\mathfrak{gl}(1|1)_1$ as follows:

$$h \cdot \Theta_1 = y_1 \Theta_1 y_2^{-1}, \quad h \cdot \Theta_2 = y_2 \Theta_2 y_1^{-1}.$$

Using the theory developed in the previous section, we now want to reconstruct the multiplication map of G in terms of the corresponding SHCP. Introduce the linear operators

$$f_i \colon \bigwedge(\mathfrak{gl}(1|1)_1) \to C^\infty(\widetilde{G}), \quad 1 \mapsto y_i, \quad \Theta_1, \Theta_2, \Theta_1 \wedge \Theta_2 \mapsto 0$$

and

$$\varphi_i \colon \bigwedge(\mathfrak{gl}(1|1)_1) \to C^\infty(\widetilde{G}), \quad \Theta_i \mapsto 1, \quad 1, \Theta_{j \neq i}, \Theta_1 \wedge \Theta_2 \mapsto 0.$$

The maps $\{f_i, \varphi_i\}$ are going to be our (global) coordinates on $\mathcal{O}(G) = \underline{\mathrm{Hom}}(\bigwedge(\mathfrak{gl}(1|1)), C^\infty(\widetilde{G}))$. They extend in a natural way to $\mathfrak{U}(\mathfrak{g}_0)$-linear maps from $\mathfrak{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$ to $C^\infty(\widetilde{G})$, which we denote by the same letter. We write $\hat{f}$ (resp. $\hat{\varphi}$) for the composition $f \circ \gamma^{-1}$ (resp. $\varphi_i \circ \gamma^{-1}$).

We want to calculate the pullbacks:

$$\begin{aligned}(\mu^*(f_i))(X, Y)(g, h) &:= \hat{f}_i(h^{-1} \cdot \gamma(X)\gamma(Y))(gh) \\ &= f_i(\hat{\gamma}^{-1}(h^{-1} \cdot \gamma(X)\gamma(Y)))(gh),\end{aligned} \tag{7.14}$$

$$\begin{aligned}(\mu^*(\varphi_i))(X, Y)(g, h) &:= \hat{\varphi}_i\big(h^{-1} \cdot \gamma(X)\gamma(Y))(gh) \\ &= \varphi_i(\hat{\gamma}^{-1}(h^{-1} \cdot \gamma(X)\gamma(Y)))(gh).\end{aligned} \tag{7.15}$$

In order to perform the computations we first need to compute the elements $\hat{\gamma}^{-1}(h^{-1} \cdot \gamma(X)\gamma(Y))$. The next table collects them.

X \ Y	$\mathbf{1}$	$\mathbf{\Theta_1}$	$\mathbf{\Theta_2}$	$\mathbf{\Theta_1 \wedge \Theta_2}$
$\mathbf{1}$	1	Θ_1	Θ_2	$\Theta_1 \wedge \Theta_2$
$\mathbf{\Theta_1}$	$y_1^{-1} y_2 \Theta_1$	0	$y_2^{-1} y_1 (\Theta_1 \wedge \Theta_2$ $-\frac{1}{2}(X_1 + X_2))$	$\frac{y_1^{-1} y_2}{2}(X_1 + X_2)\Theta_1$
$\mathbf{\Theta_2}$	$y_2^{-1} y_1 \Theta_2$	$y_2^{-1} y_1 (-\Theta_1 \wedge \Theta_2$ $-\frac{1}{2}(X_1 + X_2))$	0	$-\frac{y_2^{-1} y_1}{2}(X_1 + X_2)\Theta_2$
$\mathbf{\Theta_1 \wedge \Theta_2}$	$\Theta_1 \wedge \Theta_2$	$-\frac{1}{2}(X_1 + X_2)\Theta_1$	$\frac{1}{2}(X_1 + X_2)\Theta_2$	$\frac{1}{4}(X_1 + X_2)^2$

From this and using Definitions (7.14) and (7.15), we can calculate easily the various pullbacks. Let us do it in detail in the case of f_1. The pullback table of $(\mu^*(f_1))(X, Y)((x_1, x_2), (y_1, y_2))$ is the following.

X \ Y	$\mathbf{1}$	$\mathbf{\Theta_1}$	$\mathbf{\Theta_2}$	$\mathbf{\Theta_1 \wedge \Theta_2}$
$\mathbf{1}$	$x_1 y_1$	0	0	0
$\mathbf{\Theta_1}$	0	0	$-\frac{1}{2} x_1 y_1$	0
$\mathbf{\Theta_2}$	0	$-\frac{1}{2} y_2^{-1} x_1 y_1^2$	0	0
$\mathbf{\Theta_1 \wedge \Theta_2}$	0	0	0	$\frac{1}{4} x_1 y_1$

The link with the form of the multiplication morphism as given in equation (7.13) is established by the isomorphism

$$x_1 = f_1\big(1 + \tfrac{\varphi_1 \varphi_2}{2}\big), \quad x_2 = f_2\big(1 - \tfrac{\varphi_1 \varphi_2}{2}\big), \quad \theta_i = f_i \varphi_i .$$

The SHCP give us immediately the correspondence between Lie subgroups of a supergroup G and Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$.

Proposition 7.4.15. *Suppose that G is a connected super Lie group with super Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. There exists a unique immersed subgroup (H, j) whose Lie superalgebra is $\mathfrak{h}$.*

Proof. Since $\widetilde{G}$ is a connected Lie group, there exists a unique immersed Lie subgroup $(\widetilde{H}_0, \tilde{j})$ with $\tilde{j} : \widetilde{H} \to \widetilde{G}$. Let H denote the super Lie group associated with the super Harish-Chandra pair $(\widetilde{H}, \mathfrak{h})$, and define

$$j^* : \mathcal{O}(G) \to \mathcal{O}(H), \quad f \to \tilde{j}^*(f|_{\mathfrak{U}(\mathfrak{h})}).$$

It is immediate that j^* so defined is a superalgebra morphism, thus determining a morphism $j : H \to G$. It is a super Lie group morphism since $\tilde{j}$ is a classical Lie group morphism. It is an immersion at e (and hence everywhere), in fact we have

$$[(dj)_e X_e](\phi) = (X_e \circ j^*)(\phi) = \mathrm{ev}_e D_X^L j^*(\phi) = (D_X^L j^*)(\phi)(1)(e) = j^*(\phi)(X)(e).$$

□

7.5 Homogeneous one-parameter supergroups

In this section we want to discuss *homogeneous one-parameter supergroups* for matrix Lie supergroups. This is not the most general notion of one-parameter subgroup for a Lie supergroup since we can also have non-homogeneous ones that have to be introduced with a careful treatment of the exponential morphism. We will, however, not pursue this here (see [50] for more details).

In the ordinary setting, any element in a Lie algebra $\mathfrak{g}$ generates a one-dimensional Lie subalgebra of $\mathfrak{g}$, which is the tangent space to the one-parameter subgroup associated with such an element. This fact is no longer true for Lie superalgebras since we know that the bracket $[x,x]$ for an odd element x may not be zero, hence the Lie subalgebra generated by such an element is no longer in general one-dimensional. So it makes sense to introduce the notion of *cyclic subalgebra*.

Definition 7.5.1. Let $\mathfrak{g}$ be a real Lie superalgebra. A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called *cyclic* if it is generated by a single element $x \in \mathfrak{g}$. We then write $\mathfrak{k} = \langle x \rangle$.

As we already remarked, in the classical framework, the similar notion of cyclic Lie subalgebra is trivial: one always has $\langle x \rangle = \mathbb{R} \cdot x$ because of the identity $[x,x] = 0$.

We single out three special cases, for a given x homogeneous in $\mathfrak{g}$:

$$\begin{aligned} x \in \mathfrak{g}_0 &\implies \langle x \rangle = \mathbb{R} \cdot x, \\ x \in \mathfrak{g}_1,\ [x,x] = 0 &\implies \langle x \rangle = \mathbb{R} \cdot x, \\ x \in \mathfrak{g}_1,\ [x,x] \neq 0 &\implies \langle x \rangle = \mathbb{R} \cdot x \oplus \mathbb{R} \cdot [x,x]. \end{aligned} \tag{7.16}$$

In particular, notice that in the third case the sum is direct because $[x,x] \in \mathfrak{g}_0$, and $\mathfrak{g}_0 \cap \mathfrak{g}_1 = \{0\}$. Moreover, $\langle x \rangle = \mathbb{R} \cdot x \oplus \mathbb{R} \cdot [x,x]$ because $[x,[x,x]] = 0$ by the (super) Jacobi identity.

Notice that while the Lie superalgebra structure in the first two cases is trivial, in the third case instead, setting $y := [x,x]$, we have

$$|x| = 1, \quad |y| = 0, \quad [x,x] = y, \quad [y,y] = 0, \quad [x,y] = 0 = [y,x].$$

A one-parameter subgroup of an ordinary Lie group G is the unique (connected) subgroup K which corresponds, via the Frobenius theorem, to a specific one-dimensional Lie subalgebra $\mathfrak{k}$ of $\mathrm{Lie}(G)$. To describe such K one can use the exponential map. When $\mathfrak{g}$ is linearized and expressed by matrices, the exponential map is described by the usual formal series on matrices: $\exp(X) := \sum_{n=0}^{+\infty} X^n/n!$.

We shall now adapt this approach to the context of Lie supergroups.

Let G be a Lie supergroup and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie superalgebra. Assume further that G is embedded as a supergroup into $\mathrm{GL}(V)$ for some suitable finite-dimensional super vector space V; in other words, G is realized as a matrix Lie supergroup.

Recall that – see Definition 7.5.1 – in the super context the role of one-dimensional Lie subalgebras is played by cyclic Lie subalgebras.

Definition 7.5.2. Let $X \in \mathfrak{g} = \mathrm{Lie}(G)$ be a homogeneous element. We define the *one-parameter subgroup* associated with X to be the Lie subgroup of G corresponding to the cyclic Lie supersubalgebra $\langle X \rangle$, generated by X in $\mathfrak{g}$, via the Frobenius theorem for Lie supergroups (see Chapter 6).

For the definition it is clear that in general one-parameter supergroups may depend on more than just one parameter, however we prefer to keep this terminology in analogy with the ordinary case.

Now we describe these one-parameter subgroups. Fix a supermanifold T and set $A := \mathcal{O}(T)$ (the superalgebra of global sections). Let $t \in A_0$, $\theta \in A_1$, and $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_1$, $Z \in \mathfrak{g}_0$ such that $[Y, Z] = 0$. We define

$$\begin{aligned} \exp(tX) &:= \sum_{n=0}^{+\infty} t^n X^n/n! = 1 + tX + \tfrac{t^2}{2!} X^2 + \cdots \in \mathrm{GL}(V(T)), \qquad (7.17)\\ \exp(\vartheta Y) &:= 1 + \vartheta Y \in \mathrm{GL}(V(T)),\\ \exp(tZ + \vartheta Y) &:= \exp(tZ) \cdot \exp(\vartheta Y)\\ &= \exp(\vartheta Y) \cdot \exp(tZ)\\ &= \exp(tZ) \cdot (1 + \vartheta Y)\\ &= (1 + \vartheta Y) \cdot \exp(tZ) \in \mathrm{GL}(V(T)). \end{aligned}$$

All these expressions single out well-defined elements in $\mathrm{GL}(V(T))$. In particular, $\exp(tX)$ in (7.17) belongs to the subgroup of $\mathrm{GL}(V(T))$ whose elements are all the block matrices with zero off diagonal blocks. This is the standard group of matrices $\mathrm{GL}(A_0 \otimes V_0) \times \mathrm{GL}(A_0 \otimes V_1)$, and $\exp(tX)$ is defined here as the usual exponential of a matrix: in particular, no convergence problems arise.

The set $\exp(A_0 X) = \{\exp(tX) \mid t \in A_0\}$ is clearly a subgroup of $\mathrm{GL}(V(T))$, once we define, very naturally, the multiplication as

$$\exp(tX) \cdot \exp(sX) = \exp((t + s)X).$$

On the other hand, if we consider the same definition for $\exp(A_1 Y) := \{\exp(\vartheta Y) \mid \vartheta \in A_1\}$, we notice that in general it is not a subgroup. In fact, if we compute

$$\exp(\vartheta_1 Y) \cdot \exp(\vartheta_2 Y) = (1 + \vartheta_1 Y)(1 + \vartheta_2 Y) = 1 + \vartheta_1 Y + \vartheta_2 Y + \vartheta_1 \vartheta_2 Y^2,$$

it follows that it is not the same as

$$\exp((\vartheta_1 + \vartheta_2)Y) = 1 + (\vartheta_1 + \vartheta_2)Y = 1 + \vartheta_1 Y + \vartheta_2 Y.$$

So, recalling that $Y^2 = [Y, Y]/2$, we find that $\exp(A_1 Y)$ is a subgroup if and only if $[Y, Y] = 0$ or $\vartheta_1 \vartheta_2 = 0$ for all $\vartheta_1, \vartheta_2 \in A_1$. This reflects the fact that the $\mathbb{R}$-span of $X \in \mathfrak{g}_0$ is always a Lie supersubalgebra of $\mathfrak{g}$, but the $\mathbb{R}$-span of $Y \in \mathfrak{g}_1$ is a Lie supersubalgebra if and only if $[Y, Y] = 0$.

Thus, when $[Y, Y] \neq 0$ we must take $\exp(\langle Y \rangle(T)) = \exp(A_1 Y + A_0 Y^2)$ as the one-parameter subgroup corresponding to the Lie supersubalgebra $\langle Y \rangle$.

We summarize our discussion in the following result.

Proposition 7.5.3. *There are three distinct types of one-parameter subgroups associated with a homogeneous element in* $\mathfrak{g}$ *of dimension* $1|0$, $0|1$, *and* $1|1$, *respectively. Their functors of points are:*

(a) *for any* $X \in \mathfrak{g}_0$ *we have*

$$x_X(T) = \{\exp(tX) \mid t \in \mathcal{O}(T)_0\} = \mathbb{R}^{1|0}(T) = \mathrm{Hom}(C^\infty(x), \mathcal{O}(T));$$

(b) *for any* $Y \in \mathfrak{g}_1$, $[Y, Y] = 0$, *we have*

$$x_Y(T) = \{\exp(\vartheta Y) = 1 + \vartheta Y \mid \vartheta \in \mathcal{O}(T)_1\} = \mathbb{R}^{0|1}(T) = \mathrm{Hom}(\mathbb{R}[\xi], \mathcal{O}(T));$$

(c) *for any* $Y \in \mathfrak{g}_1$, $Y^2 := [Y, Y]/2 \neq 0$, *we have*

$$\begin{aligned} x_Y(T) &= \{\exp(tY^2 + \vartheta Y) \mid t \in \mathcal{O}(T)_0,\ \vartheta \in \mathcal{O}(T)_1\} \\ &= \mathbb{R}^{1|1}(T) = \mathrm{Hom}(C^\infty(x)[\xi], \mathcal{O}(T)). \end{aligned}$$

In cases (a) *and* (b) *the multiplication structure is trivial, while in case* (c) *it is given by* $(t, \vartheta) \cdot (t', \vartheta') = (t + t' - \vartheta\vartheta', \vartheta + \vartheta')$.

Proof. The case (a), namely when X is even, is clear. When instead X is odd we have two possibilities: either $[X, X] = 0$ or $[X, X] \neq 0$. The first possibility corresponds, by the Frobenius theorem, to a $0|1$-dimensional subgroup whose functor of points is, as is easily seen, representable and of the form (b). Let us now examine the second possibility.

The Lie subalgebra $\langle X \rangle$ generated by X is of dimension $1|1$ by (7.16). Hence by the Frobenius theorem it corresponds to a Lie subgroup of the same dimension, isomorphic to $\mathbb{R}^{1|1}$.

Now we compute the group structure on this $\mathbb{R}^{1|1}$, using the usual functor of points notation to give the operation of the supergroup. For any commutative superalgebra A, we have to calculate $t'' \in A_0$, $\vartheta'' \in A_1$ such that

$$\exp(tX^2 + \vartheta X) \cdot \exp(t'X^2 + \vartheta' X) = \exp(t''X^2 + \vartheta'' X)$$

where $t, t' \in A_0$, $\vartheta, \vartheta' \in A_1$. The direct calculation gives

$$\begin{aligned} &\exp(tX^2 + \vartheta X) \cdot \exp(t'X^2 + \vartheta' X) \\ &\quad = (1 + \vartheta X) \exp(tX^2) \cdot \exp(t'X^2)(1 + \vartheta' X) \\ &\quad = (1 + \vartheta X) \exp((t + t')X^2)(1 + \vartheta' X) \\ &\quad = \exp((t + t')X^2)(1 + \vartheta X)(1 + \vartheta' X) \\ &\quad = \exp((t + t')X^2)(1 + (\vartheta + \vartheta')X - \vartheta\vartheta' X^2) \\ &\quad = \exp((t + t')X^2)(1 - \vartheta\vartheta' X^2)(1 + (\vartheta + \vartheta')X) \\ &\quad = \exp((t + t')X^2) \exp(-\vartheta\vartheta' X^2)(1 + (\vartheta + \vartheta')X) \\ &\quad = \exp((t + t' - \vartheta\vartheta')X^2)(1 + (\vartheta + \vartheta')X) \\ &\quad = \exp((t + t' - \vartheta\vartheta')X^2 + (\vartheta + \vartheta')X), \end{aligned}$$

where we use the property

$$\exp(U+V) = \exp(U)\exp(V) \quad \text{if } [U,V]=0,$$

and also the fact that $[X, X^2] = [X^2, X^2] = 0$. □

7.6 References

Historically the concept of supermanifold was introduced in order to formalize the notion of super Lie group. In fact, although super Lie algebras were already an object of study by the physicists who originally introduced them to describe super-time infinitesimal symmetries (see [36], [80], [34], [65]), the geometrical global objects encoding such infinitesimal structure were not introduced until 1975 in [11].

The basic reference where super Lie groups are systematically studied for the first time is again [49]. In particular, in [49], Kostant introduced Lie–Hopf algebras and stated the analogue of Theorem 7.4.5. In the notes by Deligne and Morgan [22] there is a brief discussion of super Harish-Chandra pairs that we fully develop in Section 7.4. The realization of the structure sheaf of a super Lie group we give in terms of the corresponding super Harish-Chandra pair is due to Koszul and appeared in [50].

The material in Section 7.5 appeared first, to our knowledge, in [31].

8

Actions of super Lie groups

In this chapter, we want to focus our attention on the concept of action of a super Lie group G on a supermanifold M. When G acts on M, if we fix a topological point $p \in |M|$, the orbit morphism $G(T) \ni g \mapsto g \cdot p \in M(T)$ is a *constant rank morphism*. This nice property gives us the representability of the stabilizer functor and hence allows us to show right away the representability of all the supergroup functors for the classical supergroups like $\mathrm{SL}_{m|n}$ and $\mathrm{Osp}_{m|n}$, the orthosymplectic supergroup (see Example 8.4.8).

In the previous chapter we have seen how the concept of super Lie group is essentially captured by the super Harish-Chandra pair associated with it, so that we have an equivalence of categories between super Lie groups and SHCP. It is only natural to ask how the concept of action of a super Lie group translates in the language of SHCP, to give an equivalent approach to the theory of homogeneous spaces.

8.1 Actions of super Lie groups on supermanifolds

Let us start by briefly recalling the notion of action of an ordinary Lie group on an ordinary manifold M.

Definition 8.1.1. Let G be a Lie group, M a manifold. We say that G *acts* on M or equivalently that M is a G*-space* if there is a morphism $G \times M \to M$, $(g, x) \mapsto g \cdot x$, such that

(1) $1 \cdot x = x$ for all $x \in M$,

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in M$ and all $g_1, g_2 \in G$.

In Chapter 7 we have briefly discussed the definition and a few properties of actions of Lie supergroups on a supermanifold. Let us quickly recall them.

Definition 8.1.2. Let G be a super Lie group, with multiplication, inverse and unit given by μ, i and e, respectively. A morphism of supermanifolds

$$a\colon G \times M \to M$$

is called a (left) *action* of G on M if it satisfies for all supermanifolds T:

(1) $1 \cdot x = x$ for all $x \in M(T)$, 1 the unit in $G(T)$,

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in M(T)$ and all $g_1, g_2 \in G(T)$,

or equivalently:

$$a \circ (\mu \times \mathrm{id}_M) = a \circ (\mathrm{id}_G \times a), \tag{8.1a}$$

$$a \circ \langle \hat{e}, \mathrm{id}_M \rangle = \mathrm{id}_M. \tag{8.1b}$$

If we have an action a of G on M, then we say that G acts on M, or that M is a *G-supermanifold*.

Remark 8.1.3. Equations (8.1a) and (8.1b) correspond to the commutativity of the diagrams

$$\begin{array}{ccc} G \times G \times M & \xrightarrow{\mu \times \mathrm{id}_M} & G \times M \\ \downarrow{\scriptstyle \mathrm{id}_G \times a} & & \downarrow{\scriptstyle a} \\ G \times M & \xrightarrow{a} & M, \end{array} \qquad \begin{array}{ccc} M \simeq \mathbb{R}^{0|0} \times M & \xrightarrow{e \times \mathrm{id}} & G \times M \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle a} & \\ & M & \end{array}$$

and formalize the usual properties satisfied by classical actions. As we already pointed out in Remark 7.2.8, we could also define right actions by introducing the flip morphism in equation (8.1a).

Let $p \in |M|$. We can interpret, as usual, a point p in the topological space $|M|$ as an element of $M(\mathbb{R}^{0|0})$, i.e., as morphism $p_{\mathbb{R}^{0|0}} \colon \mathbb{R}^{0|0} \to M$ with $|p_{\mathbb{R}^{0|0}}| \colon \mathbb{R}^0 \to |M|$ assigning to the only element in $\mathbb{R}^0$ the point p and $p^*_{\mathbb{R}^{0|0}}$ being the evaluation at p. Since we always have the unique morphism $T \to \mathbb{R}^{0|0}$ for any supermanifold T, by functoriality we also have a morphism $M(\mathbb{R}^{0|0}) \to M(T)$ and we can define p_T the image of $p_{\mathbb{R}^{0|0}}$ under such a morphism. We also define $\hat{p} \colon M \to M$ as the morphism $\hat{p}_T \colon M(T) \to M(T), m \mapsto p_T$.

Definition 8.1.4. If $p \in |M|$ and $g \in |G|$ define the morphisms

$$a_p \colon G \to M, \quad a^g \colon M \to M$$

in the functor of points notation as

$$a_p \colon G(S) \to M(S), \quad g \mapsto g \cdot p_S,$$
$$a^g \colon M(S) \to M(S), \quad m \mapsto g_S \cdot m.$$

Equivalently:

$$a_p := a \circ \langle \mathrm{id}_G, \hat{p} \rangle, \quad a^g := a \circ \langle \hat{g}, \mathrm{id}_M \rangle,$$

$$\begin{array}{ccc} G \simeq G \times \mathbb{R}^{0|0} & \xrightarrow{\mathrm{id}_G \times p} & G \times M \\ & {\scriptstyle a_p} \searrow \quad \swarrow {\scriptstyle a} & \\ & M, & \end{array} \qquad \begin{array}{ccc} M \simeq \mathbb{R}^{0|0} \times M & \xrightarrow{g \times \mathrm{id}_M} & G \times M \\ & {\scriptstyle a^g} \searrow \quad \swarrow {\scriptstyle a} & \\ & M. & \end{array}$$

The maps a_p and a^g satisfy the relations

- $a^g \circ a^{g^{-1}} = \mathrm{id}_M$ for all $g \in |G|$,
- $a^g \circ a_p = a_p \circ \ell_g$ for all $g \in |G|$ and $p \in |M|$.

We call a_p the *orbit morphism.*

The next proposition states some nice properties of the maps a^g and a_p. In particular we have that a_p is a *constant rank morphism*. In the super context, this is a more delicate result than its classical counterpart since, as we have seen in Chapter 5, the concept of constant rank itself is more subtle. As we shall see, this result is of fundamental importance in the proof of the existence of the stabilizer subgroup (see Section 8.4).

Proposition 8.1.5. *Let G be a Lie supergroup acting on a supermanifold M via the action a. Then*

(1) *a^g is a superdiffeomorphism for all $g \in |G|$,*

(2) *a_p has constant rank for all $p \in |M|$.*

Proof. The first item follows at once from $a^g \circ a^{g^{-1}} = \mathrm{id}_M$ for all $g \in |G|$.

Let us consider the second point. Suppose that M is a supermanifold of dimension $(\hat{m}, \hat{n})$ and G is a super Lie group of dimension $(\hat{k}, \hat{q})$.

Let $\mathfrak{g}$ be the super Lie algebra of G and let J_{a_p} be the Jacobian matrix of a_p in a neighbourhood of a point $g \in |G|$. Since

$$|J_{a_p}|(g) = (\mathrm{d}a_p)_g = (\mathrm{d}a^g)_p (\mathrm{d}a_p)_e (\mathrm{d}\ell_{g^{-1}})_g$$

and a^g and $\ell_{g^{-1}}$ are diffeomorphisms, $|J_{a_p}|(g)$ has rank $\dim \mathfrak{g} - \dim\ker(\mathrm{d}a_p)_e$ for each $g \in |G|$. Recall that if $X \in \mathfrak{g}$ we denote by $D_X^L := (\mathbb{1} \otimes X)\mu^*$ the left-invariant vector field associated with X. Using equation (8.1a) we have, for each $X \in \ker(\mathrm{d}a_p)_e$,

$$\begin{aligned} D_X^L a_p^* &= (\mathbb{1} \otimes X)\mu^*(\mathbb{1} \otimes \mathrm{ev}_p)a^* \\ &= (\mathbb{1} \otimes X \otimes \mathrm{ev}_p)(\mu^* \otimes \mathbb{1})a^* \\ &= (\mathbb{1} \otimes (\mathrm{d}a_p)_e(X))a^* = 0. \end{aligned} \tag{8.2}$$

If $\{t^i, \theta^j\}$ and $\{x^k, \xi^l\}$ are coordinates in a neighbourhood U of e, and in a neighbourhood $V \supseteq |a_p|(U)$ of p, respectively, then

$$J_{a_p} = \begin{pmatrix} \frac{\partial a_p^*(x^k)}{\partial t^i} & -\frac{\partial a_p^*(x^k)}{\partial \theta^j} \\ \frac{\partial a_p^*(\xi^l)}{\partial t^i} & \frac{\partial a_p^*(\xi^l)}{\partial \theta^j} \end{pmatrix} \in \mathrm{M}_{\hat{m},\hat{n}|\hat{k},\hat{q}}(\mathcal{O}_G(U)).$$

We want to find a matrix $A \in \mathrm{GL}_{\hat{k}|\hat{q}}(\mathcal{O}_G(U))$ such that $J_{a_p}A$ has a certain set of column equal to zero. We are going to use equation (8.2).

Let $m|n = \dim\ker(\mathrm{d}a_p)_e$ and let $\{X_u\}$ and $\{\Xi_v\}$ be bases of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ such that $X_u, \Xi_v \in \ker(\mathrm{d}a_p)_e$ for $u \leq m$ and $v \leq n$. Let

$$D_{X_u}^L = \sum_i a_{u,i}\frac{\partial}{\partial t^i} + \sum_j \beta_{u,j}\frac{\partial}{\partial \theta^j}, \quad D_{\Xi_v}^L = \sum_i \gamma_{v,i}\frac{\partial}{\partial t^i} + \sum_j d_{v,j}\frac{\partial}{\partial \theta^j}$$

(with $a_{u,i}, d_{v,j} \in \mathcal{O}_G(U)_0$ and $\beta_{u,j}, \gamma_{v,i} \in \mathcal{O}_G(U)_1$) and $A = \left(\begin{smallmatrix} a_{u,i} & -\gamma_{v,i} \\ \beta_{u,j} & d_{v,j} \end{smallmatrix}\right)$. Clearly, the vector fields X_u, Ξ_v being linearly independent at each point, we have that the reduced matrix $|A|$ is invertible so that $A \in \mathrm{GL}_{\hat{k}|\hat{q}}(\mathcal{O}_G(U))$.

Now use equation (8.2) to conclude that the matrix

$$J_{a_p} A = \begin{pmatrix} D^L_{X_u} a^*_p(x^k) & -D^L_{\Xi_v} a^*_p(x^k) \\ D^L_{X_u} a^*_p(\xi^l) & D^L_{\Xi_v} a^*_p(\xi^l) \end{pmatrix}$$

has $m + n$ zero columns.

Suppose hence that we have an even matrix J_{a_p} in $M_{\hat{m},\hat{n}|\hat{k},\hat{q}}$ such that $|J_{a_p}|$ has rank equal to $\hat{k} - m|\hat{q} - n$ with entries in $\mathcal{O}_G(U)$ and only the first $\hat{k} - m + \hat{q} - n$ columns non-zero: $\left(\begin{smallmatrix} z & 0 \\ w & 0 \end{smallmatrix}\right)$. It is not restrictive to assume that z invertible so that the matrix $G = \left(\begin{smallmatrix} z^{-1} & 0 \\ -wz^{-1} & I \end{smallmatrix}\right)$ is such that $GJ = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. More precisely, it has the form

$$\begin{pmatrix} 0 & \alpha_1 & 0 & \beta_1 \\ 0 & \alpha_2 & 0 & \beta_2 \\ 0 & \gamma_1 & 0 & \delta_1 \\ 0 & \gamma_2 & 0 & \delta_2 \end{pmatrix}.$$

Since $|J_{a_p}|$ has rank $(\hat{k} - m, \hat{q} - n)$, we can suppose that α_1 and δ_1 are invertible. As in Proposition 5.2.6, we can rearrange the matrix so that it takes the form $\left(\begin{smallmatrix} z & 0 \\ w & 0 \end{smallmatrix}\right)$, with z invertible and the matrix $G = \left(\begin{smallmatrix} z^{-1} & 0 \\ -wz^{-1} & I \end{smallmatrix}\right)$ such that $GJ = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

We can then conclude that J_{a_p} has constant rank in U and, by translation, in all of $|G|$. □

8.2 Infinitesimal actions

In this section we discuss the infinitesimal interpretation of a super Lie group action on a supermanifold. In particular we will show that any action (resp. right action) of a Lie supergroup G on a supermanifold M induces an anti-morphism (resp. morphism) from the Lie superalgebra of G to the tangent bundle of the supermanifold.

Let G be a super Lie group, M a supermanifold, and let

$$a\colon G \times M \to M$$

be an action. If $v \in T_e G$ and U_e is a neighbourhood of the identity e, then the composition

$$\mathcal{O}_M(U) \xrightarrow{a^*} \mathcal{O}_{G\times M}(U_e \times U) \xrightarrow{(v\otimes \mathbb{1})} \mathcal{O}_M(U)$$

is a derivation of $\mathcal{O}_M(U)$ for any open $U \subset |M|$. The Leibniz property can be verified directly by calculating in local coordinates. The next definition is thus well posed.

Definition 8.2.1. The composition $(v \otimes \mathbb{1}) \circ a^*$ defines a vector field on M which we denote by $\rho_a(v)$:

$$\rho_a(v)(f) = (v \otimes \mathbb{1})(a^*(f)) \tag{8.3}$$

for f a local section on M.

It is clear that as v varies we get a well-defined morphism of super vector spaces

$$\rho_a \colon T_e(G) \to \mathrm{Vec}_M, \quad v \mapsto \rho_a(v),$$

from $T_e(G)$ into the super vector space of vector fields on M.

Remark 8.2.2. Let $M = G$ and let the map $a = \iota$ be the anti-group law defined by $\iota\colon G(S) \times G(S) \to G(S)$, $\iota(g(S), h(S)) := h(S)g(S)$. It is a right action. Then

$$\rho_\iota(v) = (v \otimes \mathbb{1})\iota^* = (\mathbb{1} \otimes v)\mu^* = D_v^L$$

is the unique left-invariant vector field on G which defines the tangent vector v at e. If we take $a = \mu$, where $\mu(gg') = gg'$ is the ordinary group law, then

$$\rho_\mu(v) = (v \otimes \mathbb{1})\mu^* = D_v^R,$$

where D_v^R is the unique right-invariant vector field defining the tangent vector v at e. We know that $v \mapsto D_v^L$ is a linear isomorphism of $T_e(G)$ with $\mathrm{Lie}(G)$, and one can check that $D_v^L \mapsto D_v^R$ is an anti-isomorphism of super Lie algebras. We leave this to the reader as an exercise.

The next theorem asserts that the association

$$\rho_a \colon D_v^L \mapsto \rho_a(v)$$

is an antimorphism of super Lie algebras from $\mathrm{Lie}(G)$ to Vec_M.

Theorem 8.2.3. *Let a be an action of the super Lie group G on the supermanifold M. The map*

$$\rho_a \colon \mathrm{Lie}(G) \to \mathrm{Vec}_M, \quad v \mapsto (v \otimes \mathbb{1})a^*,$$

(*see* (8.3)) *is an antimorphism of super Lie algebras. Moreover,*

$$(D_v^R \otimes \mathbb{1}_{\mathcal{O}(M)})(a^* f) = a^*(\rho_a(v) f) \tag{8.4}$$

for $v \in T_e(G)$, D_v^R its corresponding right-invariant vector field, and f a function on M.

Proof. In order to prove the first statement it is enough to prove the second one. Indeed, suppose that we have proved (8.4). Then

$$\begin{aligned}
\rho_a([D_v^R, D_w^R]_e) &= [(\mathrm{ev}_e \otimes \mathbb{1}) \circ ([D_v^R, D_w^R] \otimes \mathbb{1})] \circ a^* \\
&= [(\mathrm{ev}_e \otimes \mathbb{1})((D_v^R \otimes \mathbb{1})(D_w^R \otimes \mathbb{1}) - (D_w^R \otimes \mathbb{1})(D_v^R \otimes \mathbb{1}))]a^* \\
&= (\mathrm{ev}_e \otimes \mathbb{1})[(D_v^R \otimes \mathbb{1}) \circ a^* \rho_a(w) - (D_w^R \otimes \mathbb{1}) \circ a^* \rho_a(v)] \\
&= [\rho_a(v), \rho_a(w)],
\end{aligned}$$

and the first statement follows from the fact that the assignment $D^L_X \to D^R_X$ is a super Lie algebra anti-isomorphism.

It thus remains to prove equation 8.4. This is proved by

$$\begin{aligned}(D^R_v \otimes \mathbb{1})a^*(f) &= (v \otimes \mathbb{1}_{\mathcal{O}(G)} \otimes \mathbb{1}_{\mathcal{O}(M)}) \circ (\mu^* \otimes \mathbb{1}_{\mathcal{O}(M)}) \circ a^*(f)\\ &= (v \otimes \mathbb{1}_{\mathcal{O}(G)} \otimes \mathbb{1}_{\mathcal{O}(M)}) \circ (\mathbb{1}_{\mathcal{O}(G)} \otimes a^*) \circ a^*(f)\\ &= (v \otimes a^*) \circ a^*(f) = (\mathbb{1} \otimes a^*)(v \otimes \mathbb{1}) \circ a^*(f)\\ &= a^*(\rho_a(v) f).\end{aligned}$$

□

Corollary 8.2.4. *The anti-morphism ρ_a extends to an associative algebra anti-morphism (which we also call ρ_a)*

$$\rho_a \colon \mathcal{U}(\mathrm{Lie}(G)) \to \mathcal{U}(\mathrm{Vec}_M).$$

Proof. We use the universal property of the universal enveloping algebra and extend the anti-morphism by mapping basis to basis. We can characterize the extension also by the relation (8.4): for $v_1, v_2, \dots, v_k \in T_e(G)$ and f a local section of $\mathcal{O}_M$, we have

$$\begin{aligned}(v_1 v_2 \dots v_k \otimes \mathbb{1}_{\mathcal{O}(G)})(a^* f) &= a^*(\rho_a(v_1 v_2 \dots v_k) f)\\ &= a^*(\rho_a(v_1)\rho_a(v_2)\dots\rho_a(v_k) f).\end{aligned}$$

□

8.3 Actions of super Harish-Chandra pairs

In this section we want to translate the concept of action of a super Lie group on a supermanifold into the language of SHCP. As we shall see, the following definition comes naturally.

Definition 8.3.1. We say that an SHCP $(G_0, \mathfrak{g})$ *acts* on a supermanifold M if there is a pair $(\underline{a}, \rho)$ such that the following holds:

(i) $\underline{a} \colon G_0 \times M \to M$ is an action of G_0 on M;

(ii) $\rho \colon \mathfrak{g} \to \mathrm{Vec}(M)$ is a super Lie algebra anti-morphism

such that

$$\begin{aligned}&\rho|_{\mathfrak{g}_0}(X) \simeq (X \otimes \mathbb{1}_{\mathcal{O}(M)})\underline{a}^* \quad \text{for all } X \in \mathfrak{g}_0,\\ &\rho(g \cdot Y) = (\underline{a}^{g^{-1}})^* \rho(Y) (\underline{a}^g)^* \quad \text{for all } g \in |G|,\ Y \in \mathfrak{g}.\end{aligned}$$

Since the category of super Lie group is equivalent to the category of SHCP, one could ask whether assigning an SHCP action on a supermanifold M is equivalent to assigning an action of the corresponding super Lie group on M. The answer is positive as we shall presently see.

Proposition 8.3.2. *Let $a\colon G \times M \to M$ be an action of the super Lie group G on the supermanifold M, then the morphisms*

(1) $\underline{a}\colon \widetilde{G} \times M \to M$, *with* $\underline{a} := a \circ (j \times \mathrm{id}_M)$, *where* $j : \widetilde{G} \to G$ *is the natural immersion of the reduced Lie group* $\widetilde{G}$ *into* G,

$$(2)\ \rho_a\colon \mathfrak{g} \to \mathrm{Vec}(M)^{\mathrm{op}},\ X \mapsto (X \otimes \mathbb{1}_{\mathcal{O}(M)})a^* \tag{8.5}$$

define an action of the associated SHCP $(\widetilde{G}, \mathfrak{g})$ *on* M.

Proof. This is a simple check that we leave to the reader as an exercise. □

We are going to discuss the converse of the previous proposition. In other words, we want to understand how to recover the action of a super Lie group G on a supermanifold M from the knowledge of the action of its associated SHCP.

In analogy with the classical case, one could use the super Frobenius theorem to reconstruct a local action of G from its infinitesimal action (8.5). However, we want to take a different path and avoid the use of the super Frobenius theorem, proceeding instead to the explicit reconstruction of the global action.

Assume that we have an action a of G on M, and let $(\underline{a}, \rho_a)$ be the corresponding action of its SHCP as in Proposition 8.3.2. If $f \in \mathcal{O}(M)$, then by definition

$$a^*(f) \in [\underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty(\widetilde{G}))]\widehat{\otimes}\mathcal{O}(M) \simeq \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty(\widetilde{G})\widehat{\otimes}\mathcal{O}(M)).$$

Using equation (8.1a) and the form of the left-invariant vector fields as given in the previous chapter, we have

$$\begin{aligned} a^*(f)(X) &= (-1)^{|X|}[(D^L_X \otimes \mathbb{1})a^*(f)](1) \\ &= (-1)^{|X|}[(\mathbb{1} \otimes X \otimes \mathbb{1}) \circ (\mu^* \otimes \mathbb{1}) \circ a^*(f)](1) \\ &= (-1)^{|X|}[(\mathbb{1} \otimes X \otimes \mathbb{1}) \circ (\mathbb{1} \otimes a^*) \circ a^*(f)](1) \\ &= (-1)^{|X|}(\mathbb{1} \otimes \rho_a(X))(a^*(f)(1)) \\ &= (-1)^{|X|}(\mathbb{1} \otimes \rho_a(X))\underline{a}^*(f). \end{aligned}$$

This suggests the definition of the action a_ρ and proves its uniqueness, as we shall see in the next proposition.

Proposition 8.3.3. *Let $(\widetilde{G}, \mathfrak{g})$ be the SHCP associated with the super Lie group G and let $(\underline{a}, \rho)$ be an action of $(\widetilde{G}, \mathfrak{g})$ on a supermanifold M. Then there is a unique action $a_\rho\colon G \times M \to M$ of the super Lie group G on M whose reduced and infinitesimal actions are $(\underline{a}, \rho)$. a_ρ is explicitly given by*

$$\begin{aligned} a^*_\rho\colon \mathcal{O}(M) &\to \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty(\widetilde{G})\,\widehat{\otimes}\,\mathcal{O}(M)), \\ f &\mapsto [X \mapsto (-1)^{|X|}(\mathbb{1}_{C^\infty(\widetilde{G})} \otimes \rho(X))\underline{a}^*(f)]. \end{aligned} \tag{8.6}$$

Proof. Let us check that $a^*_\rho(f)$ is $\mathfrak{U}(\mathfrak{g}_0)$-linear. For all $X \in \mathfrak{U}(\mathfrak{g})$ and $Z \in \mathfrak{g}_0$ we have, using the fact that ρ_a is an anti-homomorphism,

$$\begin{aligned} a^*_\rho(f)(ZX) &= (-1)^{|X|}(\mathbb{1} \otimes \rho(ZX))\underline{a}^*(f) \\ &= (-1)^{|X|}(\mathbb{1} \otimes \rho(X))(\mathbb{1} \otimes Z_e \otimes \mathbb{1})(\mathbb{1} \otimes \underline{a}^*)\underline{a}^*(f) \\ &= (-1)^{|X|}(\mathbb{1} \otimes \rho(X))(\mathbb{1} \otimes Z_e \otimes \mathbb{1})(|\mu|^* \otimes \mathbb{1})\underline{a}^*(f) \\ &= (\widetilde{D^L_Z} \otimes \mathbb{1})[a^*_\rho(f)(X)]. \end{aligned}$$

We now check that a^*_ρ is a superalgebra morphism:

$$\begin{aligned} [a^*_\rho(f_1) \cdot a^*_\rho(f_2)](X) &= m_{C^\infty(\tilde{G})\hat{\otimes}\mathcal{O}(M)}[a^*(f_1) \otimes a^*(f_2)]\Delta(X) \\ &= (-1)^{|X|} m[(\mathbb{1} \otimes \rho(X_{(1)}))\underline{a}^*(f_1) \otimes (\mathbb{1} \otimes \rho(X_{(2)}))\underline{a}^*(f_2)] \\ &= (-1)^{|X|}(\mathbb{1} \otimes \rho(X))(\underline{a}^*(f_1) \cdot \underline{a}^*(f_2)) \\ &= a^*_\rho(f_1 \cdot f_2)(X), \end{aligned}$$

where $f_i \in \mathcal{O}(M)$ and $X_{(1)} \otimes X_{(2)}$ denote synthetically $\Delta(X)$. Concerning the "associative" property, we have, for $X, Y \in \mathfrak{U}(\mathfrak{g})$ and $g, h \in |G|$,

$$\begin{aligned} [(\mu^* \otimes \mathbb{1})a^*_\rho(f)](X,Y)(g,h) &= [a^*_\rho(f)](h^{-1} \cdot XY)(gh) \\ &= (-1)^{|X|+|Y|+|X||Y|}\rho(Y)\rho(h^{-1}.X)(\underline{a}^{gh})^*(f) \\ &= (-1)^{|X|+|Y|+|X||Y|}\rho(Y)(\underline{a}^h)^*\rho(X)(\underline{a}^g)^*(f) \\ &= [(\mathbb{1} \otimes a^*_\rho)a^*_\rho(f)](X,Y)(g,h) \end{aligned}$$

and, finally, $(\mathrm{ev}_e \otimes \mathbb{1})a^*_\rho(f) = \rho(1)(\underline{a}^e)^*(f) = f$. □

We end this section by revisiting Example 7.4.14.

Example 8.3.4. Consider again the super Lie group $G = \mathrm{GL}_{1|1}$ introduced in Example 7.4.14. Then G acts on itself by left multiplication, and, using the same notation as in the previous example, we have:

(1) Left action of $|G|$ on G:

$$\begin{pmatrix} t^1 & 0 \\ 0 & t^2 \end{pmatrix} \cdot \begin{pmatrix} x^1 & \xi_1 \\ \xi_2 & x^2 \end{pmatrix} = \begin{pmatrix} t^1x^1 & t^1\xi_1 \\ t^2\xi_2 & t^2x^2 \end{pmatrix}.$$

(2) Representation of $\mathfrak{gl}(1|1)$ on the super Lie algebra $\mathrm{Vec}(G)^{\mathrm{op}}$:

$$\begin{aligned} X_1 &\mapsto x^1\frac{\partial}{\partial x^1} + \xi_1\frac{\partial}{\partial \xi_1}, & X_2 &\mapsto x^2\frac{\partial}{\partial x^2} + \xi_2\frac{\partial}{\partial \xi_2}, \\ \Theta_1 &\mapsto x^2\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial x^1}, & \Theta_2 &\mapsto x^1\frac{\partial}{\partial \xi_2} + \xi_1\frac{\partial}{\partial x^2}. \end{aligned}$$

In this case the representation sends each element of $\mathfrak{gl}(1|1)$ into the corresponding right-invariant vector field.

The action μ can be reconstructed using equation (8.6); a simple calculation shows that

$$\mu^*(t^1) = t^1x^1(1+\theta^1\theta^2) + t^1\xi_2\theta^1, \quad \mu^*(t^2) = t^2x^2(1+\theta^1\theta^2) + t^2\xi_1\theta^2,$$
$$\mu^*(\theta^1) = t^1\xi_1(1+\theta^1\theta^2) + t^1x^2\theta^1, \quad \mu^*(\theta^2) = t^2\xi_2(1-\theta^1\theta^2) + t^2x^1\theta^2.$$

The usual form of the multiplication map (as given in Example 7.4.14) is obtained using the isomorphism

$$t^1 \mapsto t^1(1+\theta^1\theta^2), \quad \theta^1 \mapsto \frac{\theta^1}{t^1}; \quad t^2 \mapsto t^2(1+\theta^1\theta^2), \quad \theta^2 \mapsto \frac{\theta^2}{t^2}.$$

8.4 The stabilizer subgroup

Let G be a super Lie group acting on a supermanifold M. The aim of this section is to define the notion of stabilizer subgroup at a point in $|M|$ and to characterize it from different perspectives. Let us start by recalling such a concept in the ordinary setting.

Definition 8.4.1. Suppose that G is a Lie group acting on a manifold M, and let $x_0 \in M$. We define G_{x_0} the *stabilizer subgroup* at x_0 to be the subgroup of G given by

$$G_{x_0} := \{g \in G \mid g \cdot x_0 = x_0\}.$$

This definition has a natural generalization to the super context. Let $p \in |M|$ be a topological point. As we already remarked in Section 8.1, such p can be interpreted as an element $p_T \in M(T)$ for all supermanifolds T.

Definition 8.4.2. Let G be a super Lie group acting on a supermanifold M, and let $p \in |M|$. We define FG_p, the *stabilizer functor* at p, to be the functor from (smflds) to (sets) given by

$$FG_p(T) := \{g \in G(T) \mid g \cdot p_T = p_T\} \subset G(T) \quad \text{for all } T \in \text{(smflds)}.$$

(We leave to the reader its definition on morphisms.)

Notice that we have defined just a functor FG_p: (smflds) $\to$ (sets), so we are not sure that it corresponds to a supermanifold; in other words, we do not know if this is the functor of points of a supermanifold.

We want to reformulate this definition using the language of categories in order to prove the representability of such functor in the category of super Lie groups. We first need to establish some terminology.

Definition 8.4.3. Given two objects X and Y in a category and two arrows α and β between them, an *equalizer* is a universal pair (E, ϵ) that makes the following diagram commute:

$$E \xrightarrow{\epsilon} X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y .$$

This means that if $\tau \colon T \to X$ is such that $\alpha \circ \tau = \beta \circ \tau$, then there exists a unique $\sigma \colon T \to E$ such that $\epsilon \circ \sigma = \tau$. If an equalizer exists, it is unique up to isomorphism.

One can also define the notion of *coequalizer* in the same way, by reversing all the arrows.

Let $p \in |M|$ and denote by $\hat{p}$ the morphism

$$\hat{p} \colon G \xrightarrow{!} \mathbb{R}^{0|0} \xrightarrow{i_p} M,$$

where $! \colon G \to \mathbb{R}^{0|0}$ and $i_p \colon \mathbb{R}^{0|0} \to M$ denote respectively the unique morphism from G to a point supermanifold and the canonical injection of p in M.

Definition 8.4.4. Let G be a super Lie group and $a \colon G \times M \to M$ an action of G on the supermanifold M. The *stabilizer subgroup* at $p \in |M|$ is the supermanifold G_p equalizing the diagram

$$G \underset{\hat{p}}{\overset{a_p}{\rightrightarrows}} M .$$

As one can readily check, in the functor of points notation we have

$$G_p(T) = \{g \in G(T) \mid g \cdot p = p\}, \quad T \in \text{(smflds)}.$$

We notice that it is not a priori clear that such an equalizer exists. We are going to establish a proposition relating the two Definitions 8.4.2 and 8.4.4 of stabilizer functor and stabilizer subgroup, showing that the stabilizer subgroup always exists and its functor of points is precisely the stabilizer functor.

Before this, we need a category-theoretic lemma.

Lemma 8.4.5. *Let $\mathcal{C}$ be a category. The Yoneda embedding, i.e., the injection $\mathcal{C} \ni X \to \mathrm{Hom}(, X)$, preserves the equalizers.*

Proof. For this proof, we temporarily adopt the following notational convention. If X is an object in $\mathcal{C}$, the corresponding representable functor is denoted by h_X.

Suppose that $E \xrightarrow{e} X \rightrightarrows Y$ is an equalizer in a category $\mathcal{C}$. Then we need to show that for any equalizing diagram $F \to h_X \rightrightarrows h_Y$, the natural transformation $a \colon F \to h_X$ factors uniquely via $h_E \to h_X$.

For any object Z of $\mathcal{C}$ and any $x \in F(Z)$, $a_Z(x)$ is an arrow $Z \to X$ that composes equally with $X \rightrightarrows Y$ and therefore factors uniquely via e. Therefore define

$b_Z(x)\colon Z \to E$, $b_Z(x) \in h_E(Z)$ to be the unique map such that $e \circ b_Z(x) = a_Z(x)$. Such b is functorial in Z. Certainly $h_e \circ b = a$ by construction.

$$\begin{array}{ccc} F & \xrightarrow{\;a\;} & h_X \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle e} & \\ & h_E\,. & \end{array}$$

For uniqueness, if b' is some other natural transformation with $h_e \circ b' = a$, then for any object Z and any $x \in F(Z)$ we have $e \circ b'_Z(x) = a_Z(x) = e \circ b_Z(x)$. This implies that $b'_Z(x) = b_Z(x)$ since e is a monomorphism. Hence $b = b'$ and uniqueness is established. □

Lemma 8.4.6. *If A and B are two (super)algebras and α, β are morphisms between them, their coequalizer is the algebra $C = B/J$, where J is the ideal $(\alpha(a) - \beta(a) \mid a \in A)$.*

Proof. The coequalizer makes the following diagram commute:

$$C \longleftarrow B \overset{\beta}{\underset{\alpha}{\leftleftarrows}} A\,.$$

The result is then clear. □

We are ready for the main result of this section.

Proposition 8.4.7. *Let G be a Lie supergroup acting on the supermanifold M and let $p \in |M|$.*

(1) *The diagram*

$$G \overset{a_p}{\underset{\hat{p}}{\rightrightarrows}} M$$

admits an equalizer $G_p \in$ (smflds).

(2) *G_p is a super Lie subgroup of G.*

(3) *The functor $FG_p\colon T \mapsto FG_p(T)$ assigning to each supermanifold T the stabilizer of p_T of the action of $G(T)$ on $M(T)$ is represented by the supermanifold G_p.*

(4) *Let $(\widetilde{G}_p, \mathfrak{g}_p)$ be the SHCP associated with the stabilizer G_p. Then $\widetilde{G}_p \subseteq \widetilde{G}$ is the classical stabilizer of p with respect to the reduced action and $\mathfrak{g}_p = \ker da_p$.*

Proof. (1) According to Proposition 8.1.5 $a_p\colon G \to M$ is a constant rank morphism. Let $J_p = \{f \in \mathcal{O}(M) \mid f(p) = 0\}$ and I be the ideal in $\mathcal{O}(G)$ generated by $a^*(J_p)$. By Proposition 5.3.13 there exists a closed embedded submanifold $(\widehat{G}_p, j_{\widehat{G}_p})$ of G distinguished by the ideal I.

Since the embedding $j_{\widehat{G}_p} : \widehat{G}_p \to G$ is regular and closed, $j^*_{\widehat{G}_p}$ is surjective. Hence $\mathcal{O}(\widehat{G}_p) \simeq \mathcal{O}(G)/\ker j^*_{\widehat{G}_p}$, and moreover,

$$\begin{aligned}\ker j^*_{\widehat{G}_p} &= \langle a^*_p(f) \mid f \in J_p\rangle \\ &= \langle a^*_p(f - f(p)) \mid f \in \mathcal{O}(M)\rangle \\ &= \langle a^*_p(f) - \hat{p}^*(f) \mid f \in \mathcal{O}(M)\rangle.\end{aligned}$$

Therefore we have the co-equalizing diagram

$$\mathcal{O}(M) \underset{\hat{p}^*}{\overset{a^*_p}{\rightrightarrows}} \mathcal{O}(G) \xrightarrow{j^*_{\widehat{G}_p}} \mathcal{O}(\widehat{G}_p).$$

Hence $\widehat{G}_p$ is the equalizer of

$$\widehat{G}_p \xrightarrow{j_{\widehat{G}_p}} G \underset{\hat{p}}{\overset{a_p}{\rightrightarrows}} M\,.$$

By the uniqueness of the equalizer we have $\widehat{G}_p = G_p$.

(2) In order to prove (2) we have to show that G_p is a super Lie subgroup of G. Due to Yoneda's lemma, it is enough to prove (3).

(3) This can be proved easily by noticing that the functor FG_p equalizes the natural transformations

$$h_G \underset{\hat{p}}{\overset{a_p}{\rightrightarrows}} h_M\,.$$

(Again here h_G and h_M denote the functor of points of the supermanifolds G and M.) Since (see Lemma 8.4.5) the Yoneda embedding preserves equalizers and due to uniqueness, it follows that $FG_p \simeq h_{G_p}$.

(4) The first statement is clear since $|G| \simeq G(\mathbb{R}^{0|0})$ as set-theoretical groups. Moreover, since $j^*_{G_p} \circ a^*_p(f)$ is a constant for all $f \in \mathcal{O}(M)$, we have $\mathfrak{g}_p \subseteq \ker da_p$, and equality holds for dimension considerations. □

We are ready for some important examples.

Examples 8.4.8. (1) Consider the action (expressed with the functor of points notation):

$$a\colon \mathrm{GL}_{m|n} \times \mathbb{R}^{1|0} \to \mathbb{R}^{1|0}, \quad (g, c) \mapsto \mathrm{Ber}(g)c, \quad g \in \mathrm{GL}_{m|n}(T),\ c \in \mathbb{R}^{1|0}(T).$$

The stabilizer of the point $1 \in |\mathbb{R}^{1|0}|$ coincides with all the matrices in $\mathrm{GL}_{m|n}(T)$ with Berezinian equal to 1, that is, the special linear supergroup $\mathrm{SL}_{m|n}(T)$. By Proposition 8.4.7 we have immediately that $\mathrm{SL}_{m|n}$ is representable and that it is a super Lie subgroup of $\mathrm{GL}_{m|n}$.

(2) Consider the action

$$a\colon \mathrm{GL}_{m|n} \times \mathcal{B} \to \mathcal{B}, \quad (g, \psi(\cdot\,,\cdot)) \to \psi(g\,\cdot\,, g\,\cdot),$$

where $\mathcal{B}$ is the super vector space of all the symmetric bilinear forms on $\mathbb{R}^{m|n}$. Consider the point in $\mathcal{B}$ that corresponds to the standard bilinear form in $\mathcal{B}$:

$$\Phi = \begin{pmatrix} 0 & I_p & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_q \\ 0 & 0 & 0 & -I_q & 0 \end{pmatrix} \quad \text{if } m = 2p+1,\ n = 2q,$$

or

$$\Phi = \begin{pmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix} \quad \text{if } m = 2p,\ n = 2q.$$

The stabilizer of the point Φ is the supergroup functor $\mathrm{Osp}_{m|n}$. Again this is a Lie supergroup by Proposition 8.4.7.

For more details on the Lie superalgebras and the bilinear forms see Appendix A.

8.5 References

The action of the super Harish-Chandra pairs appears in [22]. The representability of the stabilizer functor in Proposition 8.4.7 is stated in [22]; however a complete proof of this statement appears only in [4].

9

Homogeneous spaces

In this chapter we examine the construction of the quotient of supergroups in the ordinary and the super geometric context. Classically we have that if G is a topological group and H is a closed subgroup, we can give to the quotient G/H the quotient topology, which is the finest topology for which the natural projection $\pi: G \to G/H$ is open. If the groups G and H have an additional structure, for example if they are Lie groups or algebraic groups or the corresponding super objects, it is very natural to ask if the topological space G/H inherits the extra structure in a unique way.

We start our discussion by describing transitive actions of super Lie groups on supermanifolds. We then proceed with a review of the classical construction of quotients of groups in the differential category. We do not include the proofs for the statements, referring the reader to [75], Ch. 2, for a comprehensive treatment of this subject. Finally we discuss the problem of the construction of quotients in the category of super Lie groups in full detail, including the description of the functor of points of quotients together with some physically relevant examples.

9.1 Transitive actions

In this section we want to describe transitive actions in the super setting and to look at them from different perspectives. Let us start by recalling the classical definition.

Let G be an ordinary Lie group.

Definition 9.1.1. We say that Lie group G acts *transitively* on an ordinary manifold M, or that M is a *homogeneous space* for G if there is an action

$$G \times M \to M, \quad g, m \mapsto g \cdot m,$$

of G on M and a point $x_0 \in M$ such that the morphism $\phi_{x_0}: G \to M$, $\phi_{x_0}(g) = g \cdot x$, is surjective.

We now turn to the supergeometric setting.

Let now G be a Lie supergroup, μ and e its multiplication and identity element respectively.

Definition 9.1.2. We say that a Lie supergroup G *acts* on a supermanifold M if there exists a morphism of supermanifolds $a: G \times M \to M$, $a(g, x) := g \cdot x$ for all $g \in G(T)$, $x \in M(T)$, where T is a generic supermanifold, such that

(1) $1 \cdot x = x$ for all $x \in M(T)$,

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in M(T)$ and all $g_1, g_2 \in G(T)$.

We say that the action a is *transitive*, if there exists $p \in |M|$ such that $a_p\colon G \to M$, $a_p(g) := g \cdot p$, is a surjective submersion. In this case we also say that M is a *homogeneous superspace* or a *G-superspace*.

Remark 9.1.3. Let $a_{g.p} = a_p \circ r_g$, with $r_g\colon G \to G$ defined by $r_g^* = (1 \otimes \mathrm{ev}_g)\mu^*$ or equivalently, in the functor of points notation, $r_g\colon G(T) \to G(T)$, $r_g(x) = xg$. If a_p is submersive for one $p \in |M|$ then it is submersive for all $p \in |M|$.

The next proposition characterizes transitive actions.

Proposition 9.1.4. *Suppose that M is a G-superspace, for a fixed action a. Then the following facts are equivalent:*

(1) *a is transitive.*

(2) *$|a|\colon |G| \times |M| \to |M|$ is transitive,*
$(\mathrm{d}a_p)_e\colon \mathfrak{g} \to T_p(M)$ is surjective for one p (hence for all $p \in |M|$).

(3) *If q denotes the odd dimension of G, then*

$$a_{p,\mathbb{R}^{0|q}}\colon G(\mathbb{R}^{0|q}) \to M(\mathbb{R}^{0|q})$$

is surjective.

(4) *The sheafification of the functor (see Appendix B) $\operatorname{im} a_p\colon (\text{smflds})^{\text{op}} \to (\text{sets})$*

$$(\operatorname{im} a_p)(T) := \{a_p \circ g \mid g \in G(T)\} = \{g \cdot p_T \mid g \in G(T)\} \subset M(T)$$

is the functor of points of M, where $p_T \in G(T)$ is the topological point $p \in |G|$ viewed as an element of $G(T)$.

Proof. (1) $\iff$ (2). This is an immediate consequence of Proposition 8.1.5 and our previous remark.

(1) $\implies$ (3). If $\phi \in M(\mathbb{R}^{0|q}) = \mathrm{Hom}(\mathbb{R}^{0|q}, M)$, let $|\phi| \in |M|$ be the image of the reduced map associated with ϕ (with a small, but usual, abuse of notation we denote by the same symbol $|\phi|$ both the morphism $\phi\colon \mathbb{R}^0 \to |M|$ and the point p which is its image $|\phi|(\mathbb{R}^0) \in |M|$). The pullback ϕ^* depends only on the restriction of the sections of $\mathcal{O}_M$ to an arbitrary neighbourhood of $|\phi|$. This is an easy exercise, which we leave to the reader. If a_p is a surjective submersion, there exists a local right inverse s of a_p defined in a neighbourhood of $|\phi|$. By the locality of ϕ, $s \circ \phi$ is a well-defined element of $G(\mathbb{R}^{0|q})$ and, moreover,

$$a_{p,\mathbb{R}^{0|q}}(s_{\mathbb{R}^{0|q}} \circ \phi) = a_{p,\mathbb{R}^{0|q}} \circ s_{\mathbb{R}^{0|q}} \circ \phi = \phi$$

so that $a_{p,\mathbb{R}^{0|q}}$ is surjective.

(3) $\implies$ (2). Suppose that $a_{p,\mathbb{R}^{0|q}}$ surjective. Looking at the reduced part of each morphism in $a_{p,\mathbb{R}^{0|0}}\big(G(\mathbb{R}^{0|q})\big)$, we have that $a_{p,\mathbb{R}^{0|0}} = |a_p|\colon |G| \to |M|$ is surjective. As a consequence of the ordinary manifold theory result in [48], Theorem 5.14, $|a|$ is a classical transitive action and $|a|_p$ is a submersion. Let now $m \in |M|$ and $\{t^i, \theta^j\}$ be coordinates in a neighbourhood U of it. Consider the element $\phi \in M(\mathbb{R}^{0|q})$ defined by

$$\phi^*\colon \mathcal{O}_M(U) \to \mathcal{O}(\mathbb{R}^{0|q}) = \bigwedge(\eta^1, \dots, \eta^q), \quad t^i \mapsto |t^i|(m), \quad \theta^j \mapsto \eta^j.$$

By surjectivity of $a_{p,\mathbb{R}^{0|q}}$, there exists $\psi \in G(\mathbb{R}^{0|q})$ such that $a_{p,\mathbb{R}^{0|q}}(\psi) = \phi$:

$$\psi^* \circ a_p^*(t^i) = |t^i|(m), \quad \psi^* \circ a_p^*(\theta^j) = \eta^j.$$

This implies that $T_m(M)_1$ is in the image of $(\mathrm{d}a_p)_{|\psi|}$. Since, by our previous considerations, $|a_p|$ is a submersion, $T_m(M)_0$ is in the image as well. Hence, due to Proposition 8.1.5 we are done.

(1) $\implies$ (4). Let us suppose that a_p is a surjective submersion. Let $m \in |M|$ and $g \in |a_p|^{-1}(m)$ ($|a_p|$ is surjective, so it exists). Since a_p is a submersion there exists $V \subseteq |G|$ with coordinates $X_1, \dots, X_{p+q}$ ($\dim G = p|q$) and $W \subseteq |M|$ with coordinates $Y_1, \dots, Y_{m+n}$ ($\dim M = m|n$) such that

$$a_p^*(Y_i) = X_i.$$

Let $t \in U \subseteq |T|$ for a generic supermanifold T and $\alpha\colon U \to M$ such that $m = |\alpha|(t)$. We can assume that $|\alpha|(U) \subseteq W$. If $\alpha^*(Y_i) = f_i \in \mathcal{O}_T(U)$, then $\beta\colon U \to V$ defined by

$$\beta^*(X_i) = \begin{cases} f_i & \text{if } i \leq m+n, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies $a_p \circ \beta = \alpha$. Then $[\alpha] \in (\operatorname{im} a_{p,T})_t$, hence $(\operatorname{im} a_{p,T})_t = M_t$. This shows that $\widetilde{\operatorname{im} a_p} = M$ (see Proposition B.2.9).

(4) $\Longleftarrow$ (1). Let us suppose that $\widetilde{\operatorname{im} a_p} = M$. Taking $T = \mathbb{R}^{0|0}$ we have that $|a_p|$ must be surjective. Let us now assume that $T = M$ and $m \in |M|$. There exists $U \ni m$ and $\psi\colon U \to G$ such that $a_p \circ \psi = \mathrm{id}_U$. Then a_p must be a submersion at $|\psi|(m)$, and this is true everywhere since a_p has constant rank. Indeed for all $g \in |G|$,

$$(\mathrm{d}a_p)_g \circ (\mathrm{d}l_g^G)_1 = (\mathrm{d}l_g^M)_{x_0} \circ (\mathrm{d}a_p)_1$$

where the isomorphisms l_g^G and l_g^M are the left actions of g on G and M, respectively. □

Remark 9.1.5. Notice that in the statement of (4) in Proposition 9.1.4, it is too restrictive to require the transitivity of a_T for each T, i.e., $\operatorname{im} a_p(T) = M(T)$. In fact, in the ordinary setting, this would imply that we can lift every morphism $T \to M$ to a morphism $T \to G$. As a consequence, we have the existence of a global section of the fibration $G \to M$ (take $T = M$ and the identity map), which is not true in general. It is hence necessary to take the sheafification of the image functor, as we do in point (4).

9.2 Homogeneous spaces: The classical construction

In this section we review the classical construction of the quotient of a complex or real Lie group G by a closed subgroup H. For the complete discussion we refer the reader to [75], p. 74.

As we shall see, this construction is made possible by the existence of a local section of the natural projection $\pi: G \to G/H$. Such a section in general does not exist in the algebraic category; this is due to the different nature of the topology involved. For this reason the arguments cannot be replicated in the algebraic setting, which requires a much deeper analysis of the problem. We shall not pursue this question further.

We have the following result.

Proposition 9.2.1. *Let G act transitively on a manifold M and let $G_{x_0} = \{g \in G \mid gx_0 = x_0\}$ be the stabilizer subgroup at $x_0 \in M$. Then G_{x_0} is a closed subgroup of G and the morphism $\phi_{x_0}: G \to M$, $\phi_{x_0}(g) = g \cdot x_0$ is a submersion of G onto M.*

Proof. See Lemma 2.9.2 in [75]. □

Let us assume that we have a Lie group G acting transitively on a manifold M and, for a fixed $x_0 \in M$, let us consider G_{x_0} the stabilizer subgroup at x_0, which is a closed subgroup of G by the previous proposition. The set G/G_{x_0} is a topological space with the quotient topology with respect to the natural projection $\pi: G \to G/G_{x_0}$. We would like to give to G/G_{x_0} a manifold structure compatible with the natural action of G. Also, we ask if every homogeneous space arises in this way and if the manifold structure on G/G_{x_0} is unique. All these questions have positive answers, summarized in the following theorem.

Theorem 9.2.2. *Let G be a Lie group, H a closed Lie subgroup. Then there exists a unique manifold structure on G/H such that the natural action of G on G/H, $g, xH \mapsto gxH$ is a morphism. Moreover, if M is any manifold on which G acts transitively and x_0 is a fixed point in M with $G_{x_0} = H$, then the map*

$$G/G_{x_0} \to M, \quad gG_{x_0} \mapsto g \cdot x_0,$$

is a diffeomorphism of G/G_{x_0} onto M.

The manifold G/H is called the *quotient* of G by H.

Proof. See Theorem 2.9.4 in [75]. □

We would like to make some comments on the construction of the manifold structure on the quotient G/H in the classical context, without going into the technicalities of Theorem 9.2.2.

We begin by defining a sheaf $\mathcal{O} = \mathcal{O}_{G/H}$ on G/H as follows. For all open subsets U in G/H, we define $\mathcal{O}(U)$ as the algebra of functions on U, consisting of all the $f: U \to \mathbb{R}$ such that $f \cdot \pi$ is C^∞ on $\pi^{-1}(U)$, where $\pi: G \to G/H$ is the natural projection. It is easy to check that this is a sheaf. We need to show that this defines a manifold structure on G/H and that G acts on the corresponding ringed space naturally. To this end, it is enough to show that, for some open subset V in G/H containing $\pi(1)$, there is a homeomorphism ξ of V with a smooth manifold W such that $\mathcal{O}|_V$ goes over $\mathcal{O}_W$ under ξ. We obtain such W as a submanifold of G containing 1 such that $V = \pi(W)$ and $\pi^{-1}(V) \cong W \times H$ in the following way. The map $w, h \mapsto wh$ of $W \times H$ into G is a diffeomorphism of $W \times H$ onto $\pi^{-1}(V)$ (which is open in G) commuting with the right actions of H ($y \mapsto yh$) on G and $(w, k) \mapsto (w, kh)$ on $W \times H$.

The crucial existence of W, based ultimately on the local Frobenius theorem, is equivalent to the existence of a local section $V \subset G/H \to W \times H \subset G$. Moreover, as mentioned above, one has $W \times H \cong WH$. The sheaf $\mathcal{O}|_V$ can hence be identified with the sheaf of C^∞ functions on W. This proves that we have defined a manifold structure on G/H. The uniqueness and the universal nature of this structure are not difficult to prove (see [75], Ch. 2).

What we have described above is the key idea to the proof of Theorem 9.2.2, and we shall see in Section 9.3 that, despite the different context, the existence of a local section plays a crucial role also in the construction of a quotient of Lie supergroups in very much the same way.

9.3 Homogeneous superspaces for super Lie groups

We are now interested in the construction of homogeneous spaces for super Lie groups.

In Section 9.2 we have seen that we have a unique manifold structure on the quotient of a Lie group by a closed subgroup, preserving the natural action of the group on its quotient. We now want to obtain a similar result in the super setting.

Let G be a Lie supergroup and H a closed Lie subgroup.[1] We want to define a supermanifold structure on the topological space $|G|/|H|$. This structure will turn out to be unique once we impose some natural conditions on the action of G on its quotient. In order to do this we first define a supersheaf $\mathcal{O}_X$ on the topological space $|X| = |G|/|H|$, in other words, we define a superspace $X = (|G|/|H|, \mathcal{O}_X)$. We then prove the local splitting property for X, that is we show that X is locally isomorphic to domain in $\mathbb{R}^{p|q}$ for some p and q. We start by defining the supersheaf $\mathcal{O}_X$ on $|G|/|H|$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. For each $Z \in \mathfrak{g}$, let D_Z be the left-invariant vector field on G defined by Z (see Chapter 7 for more details). For $x_0 \in |G|$ let ℓ_{x_0} and r_{x_0} be the left and right translations of G by x_0. We denote by $i_{x_0} = \ell_{x_0} \circ r_{x_0}^{-1}$ the inner automorphism defined by x_0. It fixes the identity and induces the transformation Ad_{x_0} on $\mathfrak{g}$ (see Definition 7.3.8).

[1]We say "subgroup" instead of the more cumbersome "subsupergroup". However, the reader must be aware that our H has a supergroup structure, inherited naturally from the one on G.

Definition 9.3.1. For any subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ we define the subsheaf $\mathcal{O}_{\mathfrak{k}}$ of $\mathcal{O}_G$ as

$$\mathcal{O}_{\mathfrak{k}}(U) = \{f \in \mathcal{O}_G(U) \mid D_Z f = 0 \text{ on } U \text{for all } Z \in \mathfrak{k}\}.$$

On the other hand, for any open subset $\widehat{W} \subset |G|$, invariant under right translations by elements of $|H|$, we define

$$\mathcal{O}_{\text{inv}}(\widehat{W}) = \{f \in \mathcal{O}_G(\widehat{W}) \mid f \text{ is invariant under } r_{x_0} \text{ for all } x_0 \in |H|\}.$$

If $|H|$ is connected we have

$$\mathcal{O}_{\text{inv}}(\widehat{W}) = \mathcal{O}_{\mathfrak{h}_0}(\widehat{W}),$$

as one can readily check by looking infinitesimally at the condition $r_{x_0}^* f = f$ for all $x_0 \in |H|$. For any open set $|W| \subset |X| = |G|/|H|$ with $|\widehat{W}| = \pi_0^{-1}(|W|)$ we put

$$\mathcal{O}_X(|W|) = \mathcal{O}_{\text{inv}}(\widehat{W}) \cap \mathcal{O}_{\mathfrak{h}}(\widehat{W}) \subset \mathcal{O}_G(\widehat{W}).$$

Clearly $\mathcal{O}_X(|W|) = \mathcal{O}_{\mathfrak{h}}(\widehat{W})$ if $|H|$ is connected. The subsheaf $\mathcal{O}_X$ is a supersheaf on $|X|$. We have thus defined a ringed superspace $X = (|X|, \mathcal{O}_X)$. Our aim is to prove that X is a supermanifold with $\mathcal{O}_X$ as its structure sheaf.

It is clear that the left action of the group $|G|$ on $|X|$ leaves $\mathcal{O}_X$ invariant and so it is enough to prove that there is an open neighborhood $|W|$ of the topological point $|\pi|(1) \equiv \bar{1}$ in $|G|/|H|$ with the property that $(|W|, \mathcal{O}_X|_{|W|})$ is a super domain, i.e., isomorphic to an open submanifold of $\mathbb{R}^{p|q}$.

We will do this using the local Frobenius theorem (see Chapter 6). Also, we identify as usual $\mathfrak{g}$ with the space of all left-invariant vector fields on G, thereby identifying the tangent space of G at every point canonically with $\mathfrak{g}$ itself.

On G we have a distribution spanned by the vector fields in $\mathfrak{h}$. We denote it by $\mathcal{D}_{\mathfrak{h}}$.

On each $|H|$-coset $x_0|H|$ we have a supermanifold structure which is a closed submanifold of G. It is an integral supermanifold of $\mathcal{D}_{\mathfrak{h}}$, i.e., the tangent space at any point is the subspace $\mathfrak{h}$ at that point. By the local Frobenius theorem there is an open neighborhood U of 1 and coordinates x_i, $1 \le i \le n$, and θ_α, $1 \le \alpha \le m$, on U such that $\mathcal{D}_{\mathfrak{h}}$ is spanned on U by $\partial/\partial x_i, \partial/\partial\theta_\alpha$ ($1 \le i \le r$, $1 \le \alpha \le s$). Moreover, from the theory on $|G|$ we may assume that the slices $L(\mathbf{c}) := \{(x_1, \dots, x_n) \mid x_j = c_j,\ r+1 \le j \le n\}$ are open subsets of distinct $|H|$-cosets for distinct $\mathbf{c} = (c_{r+1}, \dots, c_n)$. These slices are therefore supermanifolds with coordinates x_i, θ_α, $1 \le i \le r$, $1 \le \alpha \le s$. We have a submanifold W' of U defined by $x_i = 0$ with $1 \le i \le r$ and $\theta_\alpha = 0$ with $1 \le \alpha \le s$. The map $|\pi|\colon |G| \to |X|$ may be assumed to be a diffeomorphism of $|W'|$ with its image $|W|$ in $|X|$, and so we may view $|W|$ as a superdomain, say W. The map $|\pi|$ is then a diffeomorphism of W' with W. What we want to show is that $W \cong (|W|, \mathcal{O}_X|_{|W|})$.

Lemma 9.3.2. *The map*

$$W' \times H \xrightarrow{\gamma} G, \quad w, h \to wh,$$

is a super diffeomorphism of $W' \times H$ onto the open submanifold of G with reduced manifold the open subset $|W'||H|$ of $|G|$.

Proof. The map γ in question is the informal description of the map $\mu \circ (i_{W'} \times i_H)$ where i_M refers to the canonical inclusion $M \hookrightarrow G$ of a sub-supermanifold of G into G, and $\mu\colon G \times G \to G$ is the multiplication morphism of the Lie supergroup G. We shall use such informal descriptions without comment from now on.

It is classical that the reduced map $|\gamma|$ is a diffeomorphism of $|W'| \times |H|$ onto the open set $U = |W|'|H|$. This uses the fact that the cosets $w|H|$ are distinct for distinct $w \in |W'|$. It is thus enough to show that $\mathrm{d}\gamma$ is surjective at all points of $|W'| \times |H|$. For any $h \in |H|$, right translation by h (on the second factor in $W' \times H$ and simply r_h on G) is a super diffeomorphism commuting with γ and so it is enough to prove this at $(w, 1)$. If $X \in \mathfrak{g}$ is tangent to W' at w and $Y \in \mathfrak{h}$, then

$$\mathrm{d}\gamma(X, Y) = \mathrm{d}\gamma(X, 0) + \mathrm{d}\gamma(0, Y) = \mathrm{d}\mu(X, 0) + \mathrm{d}\mu(0, Y) = X + Y.$$

Hence the range of $\mathrm{d}\gamma$ is all of $\mathfrak{g}$ since from the coordinate chart at 1 we see that the tangent spaces to W' and $w|H|$ at w are transversal and span the tangent space to G at w which is $\mathfrak{g}$. This proves the lemma. □

Lemma 9.3.3. *We have*

$$\gamma^* \mathcal{O}_X|_{|W|} = \mathcal{O}_{W'} \otimes 1,$$

where $\gamma^*\colon \mathcal{O}_G \to \gamma_* \mathcal{O}_{W' \times H}$.

Proof. To ease the notation we drop the open set in writing a sheaf superalgebra, that is, we will write $\mathcal{O}_X$ instead of $\mathcal{O}_X(U)$.

We want to show that for any g in $\mathcal{O}_X|_U$, $\gamma^* g$ is of the form $f \otimes 1$ and that the map $g \mapsto f$ is bijective with $\mathcal{O}_{W'}$. Now γ^* intertwines D_Z ($Z \in \mathfrak{h}$) with $1 \otimes D_Z$ and so $(1 \otimes D_Z)\gamma^* g = 0$. Since the D_Z span all the super vector fields on $|H|$, it follows using charts that for any $p \in |H|$ we have $\gamma^* g = f_p \otimes 1$ locally around p for some $f_p \in \mathcal{O}_{W'}$. Clearly f_p is locally constant in p. Hence f_p is independent of p if $|H|$ is connected. If we do assume that $|H|$ is connected, the right invariance under $|H|$ shows that f_p is independent of p. In the other direction it is obvious that if we start with $f \otimes 1$, it is the image of an element of $\mathcal{O}_X|_U$. □

Theorem 9.3.4. *The superspace $(|X|, \mathcal{O}_X)$ is a supermanifold.*

Proof. By the previous lemmas we know that $(|X|, \mathcal{O}_X)$ is a super manifold at $\bar{1}$. The left invariance of the sheaf under $|G|$ shows this to be true at all points of $|X|$. □

We now want to describe the action of G on the supermanifold $X = (|G|/|H|, \mathcal{O}_X)$ we have constructed. Notice that in the course of our discussion we have also shown that there is a well-defined morphism $\pi : G \to X$.

Proposition 9.3.5. *There is a unique morphism $\beta\colon G \times X \to X$ such that the diagram*

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\beta} & X \end{array}$$

commutes.

Proof. Let $\alpha := \pi \circ \mu\colon G \times G \to X$. The action of $|G|$ on $|X|$ shows that such a map $|\beta|$ exists at the reduced level. So it is a question of constructing the pullback map

$$\beta^*\colon \mathcal{O}_X \to \mathcal{O}_{G\times X}$$

such that

$$(1 \times \pi)^* \circ \beta^* = \alpha^*.$$

Now π^* is an *isomorphism* of $\mathcal{O}_X$ onto the sheaf $\mathcal{O}_G$ restricted to a sheaf on X ($W \mapsto \mathcal{O}_G(|\pi|^{-1}(W))$), and so to prove the *existence and uniqueness* of β^* it is a question of proving that α^* and $(1 \times \pi)^*$ have the same image in $\mathcal{O}_{G\times G}$. It is easy to see that $(1 \times \pi)^*$ has as its image the subsheaf of sections f killed by $1 \otimes D_Z (Z \in \mathfrak{h})$ and invariant under $1 \times r_h (h \in |H|)$. It is not difficult to see that this is also the image of α^*. □

We tackle now the question of the uniqueness of X.

Proposition 9.3.6. *Let X' be a super manifold with $|X'| = |X|$ and let π' be a morphism $G \to X'$. Suppose that*

(1) *π' is a submersion,*

(2) *the fibers of π' are the supermanifolds which are the cosets of H.*

Then there is a natural isomorphism $X \simeq X'$.

Proof. Indeed, from the local description of submersions as projections it is clear that, for any open $|W| \subset |X|$, the elements of $\pi'^*(\mathcal{O}_{X'}(|W|))$ are invariant under $r_h, (h \in |H|)$ and killed by $D_Z(Z \in \mathfrak{h})$. Hence we have a natural map $X' \to X$ commuting with π and π'. This is a submersion, and by dimension considerations it is clear that this map is an isomorphism. □

We have proved the following result:

Theorem 9.3.7. *Let G be a Lie supergroup and H a closed Lie subgroup. There exist a supermanifold $X = (|G|/|H|, \mathcal{O}_X)$ and a morphism $\pi : G \to G/H$ such that the following properties are satisfied:*

(1) *The reduction of* π *is the natural map* $|\pi|\colon |G| \to |X|$.

(2) π *is a submersion.*

(3) *There is an action* β *from the left of* G *on* X *reducing to the action of* $|G|$ *on* $|X|$ *and compatible with the action of* G *on itself from the left through* π*:*

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\beta} & X. \end{array}$$

Moreover, the pair (X, π) *subject to the properties* (1)*,* (2)*, and* (3) *is unique up to isomorphism. The isomorphism between two choices is compatible with the actions, and it is also unique.*

Proof. This is an immediate consequence of the previous lemmas and propositions. □

9.4 The functor of points of a quotient supermanifold

In this section we want to understand how to write the functor of points for the quotient of a Lie supergroup G by a closed Lie subgroup H. Our intuition suggests that we write the functor $T \mapsto G(T)/H(T)$ that associates to each supermanifold T the coset space $G(T)/H(T)$. As we shall see, this is not far from the correct answer, however, as we already know giving a functor is by no means sufficient to define a supermanifold unless we can prove the functor is *representable*, i.e., it is the functor of points of a supermanifold.

Our goal is first to prove a *representability criterion* that enables us to single out among all the functors from the category of supermanifolds to the category of sets, those which are representable. This criterion is very formal, hence very similar in statement and in proof to the same result in the algebraic category, which we are going to describe in detail in Chapter 10. However, since the context and the notation here are slightly different, we feel that it is worth giving a proof in both cases, although the proofs of Theorem 9.4.3 and Theorem 10.3.7 are essentially the same.

This section is very much independent from the rest of our work and can be skipped in a first reading. We suggest that the reader compare it with Section 10.3 of Chapter 10, which deals with the representability issues in the algebraic context. Since our treatment is very formal and relies on very general categorical results and definitions, like Yoneda's lemma and the sheaf property of functors, the reader will realize that statements and proofs are very much the same in the differential and algebraic settings.

In dealing with representability issues we need to distinguish between a supermanifold X and its functor of points h_X, so in this section we shall use this convention. We start with the notion of local functor (see Appendix B for more details).

Definition 9.4.1. Let $F\colon (\text{smflds})^{\text{op}} \to (\text{sets})$ be a functor (not necessarily representable). We say that F is *local* if it has the sheaf property, in other words if we have the following. For any supermanifold T and any open covering $\{T_i\}$ of T let $\phi_i\colon T_i \hookrightarrow T$, $\phi_{ij}\colon T_i \cap T_j \hookrightarrow T_i$ be the natural immersions. If we have a family $\alpha_i \in F(T_i)$ such that $F(\phi_{ij})(\alpha_i) = F(\phi_{ji})(\alpha_j)$, then there exists a unique $\alpha \in F(T)$ such that $\phi_i(\alpha) = \alpha_i$.

Notice that this implies that when F is restricted to the category of the open sets of a fixed supermanifold, F is a sheaf in the ordinary sense.

Definition 9.4.2. Let U and F be functors from $(\text{smflds})^{\text{op}}$ to (sets). We say that a functor U is a subfunctor of F if $U(T) \subset F(T)$ for all $T \in (\text{smflds})$.

We further say that a subfunctor $U \subset F$ is an *open* subfunctor if for all supermanifolds S, T and morphisms $\alpha\colon h_T \to F$ we have $\alpha^{-1}(U(S)) = h_V(S)$, for V open in T. We say that a collection of open subfunctors U_i cover F if $\alpha\colon h_T \to F$, $\alpha^{-1}(U_i(T)) = h_{V_i}$ for all $T \in (\text{smflds})$, and the V_i cover T.

U is an *open supermanifold subfunctor* of F if U is a representable open subfunctor of F.

Theorem 9.4.3 (Representability criterion). *Let $F : (\text{smflds})^{\text{op}} \to (\text{sets})$ be a functor. Then the functor F is representable if and only if*

(1) *F is local,*

(2) *F is covered by the collection of open supermanifold functors $\{U_i\}$.*

Proof. If F is representable, $F \cong h_X$ for a supermanifold X. One can check directly that h_X has the two properties listed above, in particular a cover by open supermanifold functors is given by h_{X_α} where the X_α are open submanifolds covering X.

Now the other direction. We shall assume that $U_i = h_{X_\alpha}$ instead of $U_i \cong h_{X_\alpha}$ leaving to the reader the slightly more general case as an exercise.

Define $h_{X_{\alpha\beta}} := h_{X_\alpha} \times_F h_{X_\beta}$ (for the definition of fibered product refer to Appendix B).

The functor $h_{X_\alpha} \times_F h_{X_\beta}$ is representable, in fact by its very definition $h_{X_\alpha} \times_F h_{X_\beta} = i_\beta^{-1}(X_\alpha) = i_\alpha^{-1}(X_\beta)$. By our hypothesis (2) this is representable, hence we can write it as $h_{X_{\alpha\beta}}$, for $X_{\alpha\beta}$ an open submanifold of X_α. Notice that $X_{\alpha\beta}$ is also an open submanifold of X_β (by a slight abuse of notation we shall use the same symbol $X_{\alpha\beta}$ to denote isomorphic supermanifolds). As we shall see, $X_{\alpha\beta}$ corresponds to the intersection of the two open submanifolds X_α and X_β of the supermanifold X, that we shall construct, whose functor of points is F.

By the definition of fibered product we have the commutative diagram

$$\begin{array}{ccc} h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta} & \xrightarrow{j_{\beta,\alpha}} & h_{X_\beta} \\ \Big\downarrow{\scriptstyle j_{\alpha,\beta}} & & \Big\downarrow{\scriptstyle i_\beta} \\ h_{X_\alpha} & \xrightarrow{i_\alpha} & F. \end{array}$$

We now proceed and build the supermanifold X by gluing the supermanifolds X_α. We start by defining the underlying topological space.

As a set we define

$$|X| := \coprod_\alpha |X_\alpha|/\sim,$$

where $\sim$ is the following equivalence relation: $x_\alpha \sim x_\beta$ if and only if there exists $x_{\alpha\beta} \in |X_{\alpha\beta}|$ with $|j_{\alpha,\beta}|(x_{\alpha\beta}) = x_\alpha$, $|j_{\beta,\alpha}|(x_{\alpha\beta}) = x_\beta$ for all $x_\alpha \in |X_\alpha|$, $x_\beta \in |X_\beta|$. Here we use $j_{\alpha,\beta}$ to denote also the superscheme morphism $j_{\alpha,\beta}\colon X_{\alpha\beta} \to X_\alpha$. One can check this is an equivalence relation and that the map $\pi_\alpha\colon |X_\alpha| \hookrightarrow |X|$ is an injective map. $|X|$ becomes a topological space through the topology induced by the (open) topological spaces $|X_\alpha|$.

We now want to define a sheaf of superalgebras $\mathcal{O}_X$ on $|X|$ by using the sheaves $\mathcal{O}_{X_\alpha}$ and "gluing" them. Let U be open in $|X|$ and let $U_\alpha = \pi_\alpha^{-1}(U)$. Put

$$\mathcal{O}_X(U) := \{\{f_\alpha\} \in \textstyle\coprod_{\alpha\in I} \mathcal{O}_{X_\alpha}(U_\alpha) \mid j^*_{\beta,\gamma}(f_\beta) = j^*_{\gamma,\beta}(f_\gamma) \text{ for all } \beta,\gamma \in I\}.$$

The condition $j^*_{\beta,\gamma}(f_\beta) = j^*_{\gamma,\beta}(f_\gamma)$ simply states that to be an element of $\mathcal{O}_X(U)$, the collection $\{f_\alpha\}$ must be such that f_β and f_γ agree on the intersection of X_β and X_γ for any β and γ. Here we are again abusing the notation using $j_{\alpha\beta}$ for both the functor of points morphism and the supermanifold morphism. One can check directly that $\mathcal{O}_X$ is a sheaf of superalgebras by its very construction.

We have defined a supermanifold $X = (|X|, \mathcal{O}_X)$; in order to finish the proof, we need to show that $h_X \cong F$. We are looking for a functorial bijection between $h_X(T) = \mathrm{Hom}(T, X)$ and $F(T)$, for all $T \in$ (smflds).

We first construct a natural transformation $\rho_T\colon F(T) \to h_X(T)$.

Let $t \in F(T) = \mathrm{Hom}(h_T, F)$, by Yoneda's lemma. Consider the diagram

$$\begin{array}{ccc} h_{T_\alpha} := h_{X_\alpha} \times_F h_T & \longrightarrow & h_T \\ \Big\downarrow{\scriptstyle t_\alpha} & & \Big\downarrow{\scriptstyle t} \\ h_{X_\alpha} & \xrightarrow{\ i_\alpha\ } & F. \end{array}$$

Since $\{h_{X_\alpha}\}$ is an open cover of F, the $\{T_\alpha\}$ form an open cover of T. Since by Yoneda's lemma: $\mathrm{Hom}(h_{T_\alpha}, h_{X_\alpha}) \cong \mathrm{Hom}(T_\alpha, X_\alpha)$ we obtain a family of morphisms: $t_\alpha\colon T_\alpha \to X_\alpha \subset X$. The morphisms t_α glue together to give a morphism $t'\colon T \to X$, hence $t' \in h_X(T)$. So we define $\rho_T(t) = t'$.

Next we construct another natural transformation $\sigma_T\colon h_X(T) \to F(T)$, which turns out to be the inverse of ρ.

Assume that we have $f \in h_X(T)$, i.e., $f\colon T \to X$. Let $T_\alpha = f^{-1}(X_\alpha)$. We immediately obtain morphisms $g_\alpha\colon T_\alpha \to X_\alpha \subset F$. By Yoneda's lemma, g_α corresponds to a natural transformation $g_\alpha\colon h_{T_\alpha} \to h_{X_\alpha}$. Since F is local, the morphisms $i_\alpha \cdot g_\alpha\colon h_{T_\alpha} \to h_{X_\alpha} \to F$ glue together to give a morphism $g\colon h_T \to F$, i.e., an element $g \in F(T)$. Define $\sigma_T(f) = g$.

One can directly check that ρ and σ are indeed natural transformations inverse to each other, hence $F \cong h_X$. □

Remark 9.4.4. The representability criterion we just proved, holds as it is, also for the category of analytic supermanifolds discussed in Section 4.8. The proof is the same, and the reader is invited to check that at no point did we make any use of the fact that a sheaf has any local C^∞ property. In fact Theorem 9.4.3 is a categorical result, the reader may also consult [77], Ch. 1, for a more general setting of this statement. We are going to revisit this same proposition for the case of superschemes in Chapter 10. However, we are going to employ a slightly different category and for this reason the statement is only apparently not the same. All we say in Theorems 9.4.3 and 10.3.7 is contained in [77], where the treatment is the most general possible.

We now turn to examine the functor of points of the quotient of a Lie supergroup G by a closed subgroup H. We are interested in a characterization of the functor of points of $X = G/H$ in terms of the functor of points of the supergroups G and H.

Theorem 9.4.5. *Let G and H be supergroups as above and let $\widetilde{G/H}$ be the sheafification of the functor: $T \to G(T)/H(T)$. Then $\widetilde{G/H}$ is representable and is the functor of points of the homogeneous space supermanifold $X = G/H$ constructed above.*

Proof. In order to prove this result, we shall use the uniqueness property which characterizes the homogeneous space G/H (see Theorem 9.3.7). So we only need to prove that $\widetilde{G/H}$ is representable, i.e., $\widetilde{G/H} = h_X$ for a supermanifold X, and that X satisfies the three properties detailed in Theorem 9.3.7.

To prove that $\widetilde{G/H}$ is representable we use the criterion in Theorem 9.4.3. The fact that $\widetilde{G/H}$ has the sheaf property is clear by its very definition. So it is enough to prove there is a open supermanifold subfunctor of $\widetilde{G/H}$ around the origin (by translation we can transport such open supermanifold subfunctor to obtain a neighbourhood at every point). But this is given by h_W, with $W \cong W' \times H$ constructed as in the previous section.

We now turn to the properties (1), (2) and (3). (1) and (3) are left to the reader as an exercise. The fact π is a submersion, that is property (2), comes by looking at it in the local coordinates given by W. □

We now examine some interesting examples.

Example 9.4.6 (Coadjoint orbits of $\mathrm{SL}_{m|n}$). Consider the following action of $G = \mathrm{SL}_{m|n}$ on its Lie superalgebra $\mathfrak{g} = \mathfrak{sl}_{m|n}$:

$$\rho\colon G(T) \times \mathfrak{g}(T) \to \mathfrak{g}(T), \quad g, X \mapsto gXg^{-1}.$$

Let us fix a topological point $X_0 = \mathrm{diag}(\lambda_1, \dots, \lambda_{m+n}) \in |\mathfrak{g}|$, where the real numbers $\lambda_1, \dots, \lambda_{m+n}$ are all distinct. Then one sees immediately that the stabilizer subgroup H at X_0 is given by

$$H(T) = \left\{ \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & a_{m+n} \end{pmatrix} \right\} \subset G(T).$$

We now want to describe the orbit of X_0 as a submanifold of $\mathfrak{g}$. Consider the ideal I in $\mathcal{O}(\mathfrak{g})$ generated by the polynomial functions str^n, where $\mathrm{str}(Z)$ returns the supertrace of an element $Z \in \mathfrak{g}$. I is a regular ideal (see Proposition 5.3.5), hence it defines a closely embedded supermanifold X in $\mathfrak{g}$. We leave to the reader the routine check of properties (1), (2) and (3) in Theorem 9.3.7 so that we have $X = G/H$. One also immediately verifies that $h_X = \widetilde{G/H}$. Moreover, h_X is the functor of points of the regular coadjoint orbits of $\mathrm{SL}_{m|n}$. A similar calculation can be done to compute the regular orbits of the other classical supergroups.

9.5 The super Minkowski and super conformal spacetime

In this section we want to examine the example of $F\ell = F\ell(2|0, 2|1; 4|1)$: the analytic supermanifold of $2|0$, $2|1$ flags in the space $\mathbb{C}^{4|1}$ (for the definition of analytic supermanifolds refer to Section 4.8) and one of its real forms. This example has a special importance in physics since it is the complexification of the super conformal space. This supermanifold is the compactification of the Minkowski superspace in $4|4$ dimensions, where the Minkowski is realized as the "big cell", which is a dense open set inside the conformal superspace.

We define $F\ell(T)$ as the set consisting of all pairs $F_1 \subset F_2$, where F_1 and F_2 are locally free subsheaves of $\mathcal{O}_T^{4|1} := \mathcal{O}_{T,0} \otimes \mathbb{C}^{4|0} \oplus \mathcal{O}_{T,1} \otimes \mathbb{C}^{0|1}$ of rank $2|0$ and $2|1$, respectively. In other words, for each point $t \in |T|$, $(F_1)_t$ and $(F_2)_t$ are free submodules of $\mathcal{O}_{T,t}^{4|1}$ of rank $2|0$ and $2|1$, respectively (recall that projective modules over local superrings are free, see Appendix B). The pairs $F_1 \subset F_2$ are called *flags*. Notice that if T is a point so that $\mathcal{O}(T) = \mathbb{C}$, we have that $F_1 \subset F_2$ are super vector spaces of dimension respectively $2|0$ and $2|1$ in $\mathbb{C}^{4|1}$, thus recovering the usual definition of flag.

On $\mathcal{O}(T)^{4|1}$ there is a natural action of $G = \mathrm{GL}_{4|1}(T)$ which is inherited by $F\ell(T)$:

$$\mathrm{GL}_{4|1}(T) \times F\ell(T) \to F\ell(T), \quad g, F_1 \subset F_2 \mapsto g \cdot F_1 \subset g \cdot F_2,$$

where $g \cdot F_i$ are defined as follows. Since F_i is a locally free subsheaf of $\mathcal{O}_T^{4|1}$, it follows that $F_i(T_\alpha)$ is free for a suitable base $\{T_\alpha\}$ of open sets in T so that $g \cdot F_i(T_\alpha)$ makes sense. For instance, for F_1 it is

$$g \cdot \langle v_1, v_2 \rangle := \langle g v_1, g v_2 \rangle.$$

Thus we have defined $(g \cdot F_i)(T_\alpha)$ for a base of open sets. Since the compatibility conditions in Definition 2.2.10 are satisfied, $g \cdot F_i$ is a $\mathcal{B}$-sheaf and we can use Proposition 2.2.11 to obtain a sheaf on T that we denote by $g \cdot F_i$.

Let us fix the flag $\mathcal{F} = \{\mathcal{O}_T^{2|0} \subset \mathcal{O}_T^{2|1}\} \in F\ell(T)$. The stabilizer subgroup at $\mathcal{F}$ is the subgroup H of G given in terms of the functor of points by

$$H(T) = \left\{ \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} \\ 0 & 0 & g_{33} & g_{34} & 0 \\ 0 & 0 & g_{43} & g_{44} & 0 \\ 0 & 0 & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix} \right\} \in G(T).$$

In fact, one can check directly that H stabilizes $\mathcal{F}$ and that the functor $T \mapsto H(T)$ is the functor of points of a closed subgroup of G. Hence locally, we can identify the set $F\ell(T)$ with the set $G(T)/H(T)$ so that the sheafification of both functors coincide (see Proposition B.2.9). Since $F\ell$ is already a sheaf, by its very definition, we have that $F\ell$ is the functor of points of the quotient supermanifold G/H.

Next we want to describe the *big cell* inside $F\ell$. This is an open submanifold, which, as we shall see, has a special importance in physics since it is the complexification of the super Minkowski space.

Locally at a point $t \in |T|$, an element $F_1 \subset F_2$ in $F\ell(T)$ consists of two free submodules of $\mathcal{O}_{T,t}^{4|1}$ or rank $2|0$ and $2|1$, and is thus described by a pair of subspaces $\langle v_1, v_2 \rangle \subset \langle w_1, w_2, w_3 \rangle$ with $v_i, w_j \in \mathcal{O}_{T,t}^{4|1}$ of suitable parity. Let $U_1(T)$ and $U_2(T)$ be the set of subsheaves of $\mathcal{O}_T^{4|1}$ of ranks $2|0$ and $2|1$ expressed locally as

$$U_1(T)_t = \left\{ \left\langle \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{41} \\ v_{51} \end{pmatrix}, \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \\ v_{42} \\ v_{52} \end{pmatrix} \right\rangle \;\middle|\; \det \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \text{ invertible} \right\},$$

$$U_2(T)_t = \left\{ \left\langle \begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \\ w_{41} \\ w_{51} \end{pmatrix}, \begin{pmatrix} w_{12} \\ w_{22} \\ w_{32} \\ w_{42} \\ w_{52} \end{pmatrix}, \begin{pmatrix} w_{13} \\ w_{23} \\ w_{33} \\ w_{43} \\ w_{53} \end{pmatrix} \right\rangle \;\middle|\; \mathrm{Ber} \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{51} & w_{52} & w_{53} \end{pmatrix} \text{ invertible} \right\}.$$

In order to simplify the notation, from now on we shall write everything locally without further mention.

Using the identification $F\ell(T) \cong (G/H)(T)$, which locally amounts to having $F\ell(T) \cong G(T)/H(T)$, we can write an element in $U_1(T) \times U_2(T)$ uniquely as

$$\left(\begin{pmatrix} I \\ A \\ \alpha \end{pmatrix}, \begin{pmatrix} I & 0 \\ B & \beta \\ 0 & 1 \end{pmatrix} \right) \in U_1(T) \times U_2(T),$$

where I is the identity, A and B are (2×2)-matrices with even entries, and $\alpha = (\alpha_1, \alpha_2)$, $\beta^t = (\beta_1, \beta_2)$ are rows with odd entries in $\mathcal{O}_{T,t}$.

We define the *big cell* $U(T)$ inside $F\ell(T)$ as the pairs $(u, v) \in U_1(T) \times U_2(T)$. Notice that an element of $U_1(T)$ is inside $U_2(T)$ if

$$A = B + \beta\alpha. \tag{9.1}$$

So a flag in the big cell U is completely described by the triplet (A, α, β). We see also that U is an affine $4|4$ superspace, i.e., the functor U is representable and is the functor of points of a $4|4$ superspace. Equation (9.1) is also known as the *twistor relation* in the physics literature. Since there are no relations among A, α and β, we can take them as local coordinates in a neighbourhood of the identity in $|G|/|H|$.

In these coordinates, the flag $\mathcal{F}$ corresponding to the identity that we fixed at the beginning becomes

$$\left(\begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \approx (0, 0, 0).$$

The big cell $U_1 \times U_2$ is obtained by requiring the invertibility of the determinant $(1, 2)$ and the Berezinian $(1, 2, 5)$ (obtained by taking respectively columns $(1, 2)$ and rows $(1, 2)$ and columns $(1, 2, 5)$ and rows $(1, 2, 5)$) in the generic expression of a flag in $F\ell$. Clearly one can repeat the same argument and obtain a total of six different big cells by requiring the invertibility of the determinant and Berezinians:

$$\begin{array}{llllll} (1,2) & (1,2,5), & (1,3) & (1,3,5), & (1,4) & (1,4,5), \\ (2,3) & (2,3,5), & (2,4) & (2,4,5), & (3,4) & (3,4,5). \end{array}$$

We leave to the reader the easy check that these big cells cover the whole of $F\ell$. Hence we can apply the representability criterion in Theorem 9.4.3, thus proving that $F\ell$ is an analytic supermanifold.

We now want to write explicitly the morphism $\pi : G \to F\ell$, $\pi(g) = g \cdot \mathcal{F}$ in these coordinates and see it is a submersion. In a suitable open subset near the identity of the group we can take an element $g \in G(T)$ as

$$g = \begin{pmatrix} g_{ij} & \gamma_{i5} \\ \gamma_{5j} & g_{55} \end{pmatrix}, \quad i, j = 1, \dots, 4.$$

Then we can write an element $g \cdot \mathcal{F} \in U_1(T) \times U_2(T) \subset F\ell(T)$ as

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \\ \gamma_{51} & \gamma_{52} \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & \gamma_{15} \\ g_{21} & g_{22} & \gamma_{25} \\ g_{31} & g_{32} & \gamma_{35} \\ g_{41} & g_{42} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & g_{55} \end{pmatrix} \approx \begin{pmatrix} I \\ WZ^{-1} \\ \rho_1 Z^{-1} \end{pmatrix}, \begin{pmatrix} I & 0 \\ VY^{-1} & (\tau_2 - WZ^{-1}\tau_1)a \\ 0 & 1 \end{pmatrix},$$

where

$$\rho_1 = \begin{pmatrix} \gamma_{51} & \gamma_{52} \end{pmatrix}, \quad W = \begin{pmatrix} g_{31} & g_{32} \\ g_{41} & g_{42} \end{pmatrix}, \quad Z = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$
$$\tau_1 = \begin{pmatrix} \gamma_{15} \\ \gamma_{25} \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} \gamma_{35} \\ \gamma_{45} \end{pmatrix}, \quad d = (g_{55} - \nu Z^{-1}\mu_1)^{-1},$$
$$V = W - g_{55}^{-1}\tau_2\rho_1, \quad Y = Z - g_{55}^{-1}\tau_1\rho_1.$$

Finally the map π in these coordinates is given by

$$g \mapsto (WZ^{-1}, \rho_1 Z^{-1}, (\tau_2 - WZ^{-1}\tau_1)d).$$

At this point one can compute the super Jacobian and verify that at the identity it is surjective, so π is a submersion. This gives an equivalent and independent proof of the fact that $F\ell$ is the quotient G/H (see Theorem 9.3.7).

The subgroup of G leaving invariant the big cell is the set of matrices in G of the form

$$\begin{pmatrix} L & 0 & 0 \\ NL & R & R\chi \\ d\varphi & 0 & d \end{pmatrix},$$

with L, N, R being 2×2 even matrices, χ and odd 1×2 matrix, φ a 2×1 odd matrix and d a scalar. This is then what the physicists call the *complex Poincaré supergroup* and its action on the big cell can be written as

$$A \to R(A + \chi\alpha)L^{-1} + N, \quad \alpha \to d(\alpha + \varphi)L^{-1}, \quad \beta \to d^{-1}R(\beta + \chi).$$

If the odd part is zero, then the action reduces to the one of the classical Poincaré group on the ordinary Minkowski space.

We see that the big cell of the flag supermanifold $F\ell(2|0, 4|0; 4|1)$ can be interpreted as the complex super Minkowski space time, the flag being its *superconformal compactification*.

We now turn to the construction of the real Minkowski superspace which is extremely important in physics. We start with a real form for the supergroup G. In order to obtain such a form (see Section 4.8 for the definition) we need a natural transformation from G to its complex conjugate $\overline{G}$. Define the natural transformation (T being a supermanifold) as

$$G(T) \to \overline{G}(T), \quad g = \begin{pmatrix} D & \tau \\ \rho & d \end{pmatrix} \to g^\theta = \begin{pmatrix} D^\dagger & i\rho^\dagger \\ i\tau^\dagger & \bar{d} \end{pmatrix}.$$

Lemma 9.5.1. *We have* $(hg)^\theta = g^\theta h^\theta$.

Proof. Direct calculation. □

Remark 9.5.2. It is important to notice that we are under the following convention: if θ and ξ are odd variables, then

$$\overline{\theta\xi} = \bar{\theta}\bar{\xi}. \tag{9.2}$$

This convention is opposed to the one used most commonly in physics, namely

$$\overline{\theta\xi} = \bar{\xi}\bar{\theta},$$

but as it is explained in [22], it is the one that makes sense functorially. According to this convention, then for matrices X, Y with *odd* entries

$$(\overline{XY})^T = -(\bar{Y})^T(\bar{X})^T.$$

We are ready to define the involution ξ which gives the real form of G:

$$G(T) \xrightarrow{\xi} \bar{G}(T), \quad g \mapsto g^\xi := L(x^\theta)^{-1}L, \quad L = \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have $(hg)^\xi = h^\xi g^\xi$ and $(g^\xi)^\xi = g$, so it is a conjugation.

As one can readily check from the definitions, we have the following lemma and proposition.

Lemma 9.5.3. *The map $x \mapsto x^\xi$ is a natural transformation. It defines a ringed space involutive isomorphism $\rho\colon G \to \bar{G}$ which is $\mathbb{C}$-antilinear.*

Proposition 9.5.4. *The topological space G^ξ consisting of the points fixed by ρ has a real supermanifold structure and the supersheaf is composed of those functions $f \in \mathcal{O}_G$ such that $\overline{\xi^*(f)} = f$.*

The involution we chose has important physical properties and we invite the reader to consult [32] for a complete treatment of them.

Also, it is easy to check that it reduces to a conjugation on the Poincaré supergroup and we can compute explicitly such conjugation as follows:

$$g = \begin{pmatrix} L & 0 & 0 \\ M & R & R\chi \\ d\varphi & 0 & d \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} L^{-1} & 0 & 0 \\ -R^{-1}ML^{-1} + \chi\varphi L^{-1} & R^{-1} & -\chi d^{-1} \\ -\varphi L^{-1} & 0 & d^{-1} \end{pmatrix},$$

$$g^\xi = \begin{pmatrix} R^{\dagger^{-1}} & 0 & 0 \\ -L^{\dagger^{-1}}M^\dagger R^{\dagger^{-1}} - L^{\dagger^{-1}}\varphi^\dagger\chi^\dagger & L^{\dagger^{-1}} & -jL^{\dagger^{-1}}\varphi^\dagger \\ -j\bar{d}^{-1}\chi^\dagger & 0 & \bar{d}^{-1} \end{pmatrix}.$$

It follows that the fixed points are those that satisfy the conditions

$$L = R^{\dagger^{-1}}, \quad \chi = -j\varphi^\dagger, \quad ML^{-1} = -(ML^{-1})^\dagger - jL^{\dagger^{-1}}\varphi^\dagger\varphi L^{-1}. \tag{9.3}$$

To get a more familiar form for the reality conditions, we observe that the last equation in (9.3) can be cast as

$$M'L^{-1} \equiv ML^{-1} + \frac{1}{2} j L^{\dagger^{-1}} \varphi^\dagger \varphi L^{-1}, \quad M' = -M'^\dagger.$$

This is just an odd translation and amounts to multiplying g on the right by the group element

$$g' = \begin{pmatrix} I & 0 & 0 \\ -\frac{1}{2} jR^{-1} L^{\dagger^{-1}} \varphi^\dagger \varphi L^{-1} & I & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We want now to compute the real form of the big cell. The first thing to observe is that the real form is well defined on the quotient space G/H (the superflag), where H is the group described previously.

Notice that a point of the big cell (A, α, β) can be represented by an element of the group

$$g = \begin{pmatrix} I & 0 & 0 \\ A & I & \beta \\ \alpha & 0 & 1 \end{pmatrix}$$

since

$$g \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I \\ A \\ \alpha \end{pmatrix}, \quad g \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & \beta \\ \alpha & 1 \end{pmatrix} \approx \begin{pmatrix} I & 0 \\ A - \beta\alpha & \beta \\ 0 & 1 \end{pmatrix}.$$

We first compute the inverse

$$g^{-1} = \begin{pmatrix} I & 0 & 0 \\ -A + \beta\alpha & I & -\beta \\ -\alpha & 0 & 1 \end{pmatrix},$$

and then

$$g^\xi = \begin{pmatrix} I & 0 & 0 \\ -A^\dagger - \alpha^\dagger \beta^\dagger & I & -j\alpha^\dagger \\ -j\beta^\dagger & 0 & 1 \end{pmatrix}.$$

The element g^ξ is already in the desired form, so the real points are given by

$$A = -A^\dagger - j\alpha^\dagger \alpha, \quad \beta = -j\alpha^\dagger.$$

We can make a convenient change of coordinates,

$$A' \equiv A + \frac{1}{2} j\alpha^\dagger \alpha,$$

so the reality condition is

$$A' = -A'^\dagger,$$

and we recover the usual form for the (purely even) Minkowski space time.

9.6 References

This treatment of quotients of supergroups, via their functor of points and, equivalently, through the direct construction of the sheaf of invariant functions, is found in [32], [4], [5]. As for Example 9.4.6 see [76], [56].

10

Supervarieties and superschemes

The aim of this and the next chapter is to lay down the foundations of algebraic supergeometry, using the machinery of sheaves and schemes together with their functor of points that we have described in detail in the classical setting in the previous chapters. The presence of nilpotents in the structural sheaf of an ordinary scheme makes its generalization to supergeometry very natural and allows us to carry most definitions and results from the ordinary to the super setting, almost with no changes. There are, however, some remarkable differences that we shall point out along with our treatment. The most striking one is the rigidity of the projective superspace, which does not contain as many subvarieties as its classical counterpart. We shall, in fact, see that the Grassmannian superscheme is not in general a projective supervariety. This fact has some deep consequences, for instance, the strategy for the construction of the quotient of algebraic supergroups has to be suitably modified. We shall however not pursue this point further in the text.

We start by giving the definition of superscheme and its functor of points and we examine some important examples, including affine and projective superspaces and Grassmannian superschemes. We then describe a representability criterion, which allows us to single out among functors from the category of superalgebras to the category of sets those which are the functors of points of a superscheme. This criterion is especially useful in supergeometry since the functor of points is often the only way we can handle supergeometric objects. Taking this point of view, we discuss the infinitesimal theory in the algebraic super-setting. In particular we study superderivations and their relations with the tangent space at a rational point of a superscheme over a field, making some concrete calculations to exemplify our definitions. Our study of the infinitesimal theory is of capital importance for the theory of algebraic supergroups that we shall treat in the next chapter.

10.1 Basic definitions

In this section we give the basic definitions of algebraic supergeometry. Because we are in need of a more general setting in the next chapters we no longer assume that the ground field to be $\mathbb{R}$ or $\mathbb{C}$ and actually we are going to replace it with a commutative ring.

Let k be a commutative ring. Assume that all superalgebras are associative, commutative (i.e., $xy = (-1)^{p(x)p(y)}\, yx$) with unit and over k, unless otherwise specified.

We denote their category by (salg). For a superalgebra A, let J_A denote the ideal generated by the odd elements, i.e., $J_A = \langle A_1 \rangle_A$. Write A^r for the quotient A/J_A. We say that A^r is *reduced* or *super reduced* even if it may contain some (even) nilpotents.

In Chapter 3 we have introduced the categories of *superspaces* and of *superschemes*. Let us briefly recall these notions.

A superspace $X = (|X|, \mathcal{O}_X)$ is a topological space $|X|$ together with a sheaf of superalgebras $\mathcal{O}_X$ such that $\mathcal{O}_{X,x}$ is a local superalgebra, i.e., it has a unique two-sided maximal homogeneous ideal.

The sheaf of superalgebras $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_{X,0}$-modules, where $\mathcal{O}_{X,0}$ is the sheaf over $|X|$ defined as $\mathcal{O}_{X,0}(U) := \mathcal{O}_X(U)_0$ for all U open in $|X|$. Notice that also the sheaf $\mathcal{O}_{X,1}$, defined as $\mathcal{O}_{X,1}(U) = \mathcal{O}_X(U)_1$, is also a sheaf of $\mathcal{O}_{X,0}$-modules.

Given two superspaces $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$, a *morphism* $f : X \to Y$ of superspaces is given by a pair $f = (|f|, f^*)$ such that

(1) $|f| : X \to Y$ is a continuous map;

(2) $f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a map of sheaves of superalgebras on $|Y|$, that is, for U open in $|Y|$ there exists a family of morphisms $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(|f|^{-1}(U))$ compatible with restrictions;

(3) the map of local superalgebras $f_p^* : \mathcal{O}_{Y,|f|(p)} \to \mathcal{O}_{X,p}$ is a local morphism, i.e., sends the maximal ideal of $\mathcal{O}_{Y,|f|(p)}$ to the maximal ideal of $\mathcal{O}_{X,p}$.

A *superscheme* X is a superspace $(|X|, \mathcal{O}_X)$ such that $\mathcal{O}_{X,1}$ is a quasi-coherent sheaf of $\mathcal{O}_{X,0}$-modules. $\mathcal{O}_X$ is called the *structure sheaf* of X. A *morphism* of superschemes is a morphism of the corresponding superspaces.

For any open $U \subset |X|$ we define the superscheme $U = (|U|, \mathcal{O}_X|_{|U|})$, called an *open subscheme* in the superscheme X.

The most important example of a superscheme is given by the *spectrum of a superalgebra* A. It consists of the spectrum of the even part A_0 together with a certain sheaf of superalgebras on it. Let us see this construction in detail.

Definition 10.1.1 (The superscheme $\underline{\mathrm{Spec}}\, A$). Let A be an object of (salg). We have $\mathrm{Spec}(A_0) = \mathrm{Spec}(A^r)$ since the algebras A^r and A_0 differ only by nilpotent elements.

Let us consider $\mathcal{O}_{A_0}$ the structure sheaf of $\mathrm{Spec}(A_0)$ (see Chapter 2 for more details). The stalk of this sheaf at the prime $\mathfrak{p} \in \mathrm{Spec}(A_0)$ is the localization of A_0 at $\mathfrak{p}$. As for any superalgebra, A is a module over A_0. So, according to the classical construction detailed in Chapter 2, Section 2.5, we have a sheaf $\tilde{A}$ (that we shall denote also by $\mathcal{O}_A$) of $\mathcal{O}_{A_0}$-modules over $\mathrm{Spec}\, A_0$, with stalk

$$A_{\mathfrak{p}} = \left\{ \tfrac{f}{g} \mid f \in A,\ g \in A_0 - \mathfrak{p} \right\},$$

the localization of the A_0-module A over each prime $\mathfrak{p} \in \mathrm{Spec}(A_0)$. This localization has a unique two-sided maximal ideal which consists of the maximal ideal in the local ring $(A_{\mathfrak{p}})_0$ and the generators of $(A_{\mathfrak{p}})_1$ as A_0-module.

As one can easily check, $\tilde{A}$ (or $\mathcal{O}_A$) is a sheaf of superalgebras and $(\mathrm{Spec}\, A_0, \mathcal{O}_A)$ is a superscheme that we denote by $\underline{\mathrm{Spec}}\, A$. In fact we just showed that it is a superspace since $\mathcal{O}_{A,p}$ is local; moreover, by its very definition, we have that $\mathcal{O}_{A,1}$ is a quasi-coherent sheaf of $\mathcal{O}_{A,0}$-modules since it is a sheaf of $\mathcal{O}_{A_0} = \mathcal{O}_{A,0}$-modules.

Notice that on the basic open sets

$$U_f = \{\mathfrak{p} \in \operatorname{Spec} A_0 \mid (f) \not\subset \mathfrak{p}\}, \quad f \in A_0,$$

we have $\mathcal{O}_A(U_f) = A_f = \{a/f^n \mid a \in A\}$.

Definition 10.1.2. An *affine superscheme* is a superspace that is isomorphic to $\underline{\operatorname{Spec}}\, A$ for some superalgebra A in (salg). Such superalgebra A, by the construction of $\underline{\operatorname{Spec}}\, A$, is isomorphic to the superalgebra of the global sections of the structure sheaf of X, which we shall denote by $\mathcal{O}(X)$ (instead of the more cumbersome notation $\mathcal{O}_X(|X|)$).

An *affine algebraic supervariety* is a superspace isomorphic to $\underline{\operatorname{Spec}}\, A$ for some *affine* superalgebra A, i.e., a finitely generated superalgebra such that A/J_A has no nilpotents. We will call A the *coordinate ring* of the superscheme or the supervariety.

The next proposition tells us that superschemes have affine superschemes as local models, as it happens for the ordinary setting.

Proposition 10.1.3. *A superspace S is a superscheme if and only if it is locally isomorphic to $\underline{\operatorname{Spec}}\, A$ for some superalgebra A, i.e., for all $x \in |S|$ there exists $U_x \subset |S|$ open such that $(U_x, \mathcal{O}_S|_{U_x}) \cong \underline{\operatorname{Spec}}\, A$. (Clearly A depends on U_x.)*

Proof. Since S is a superscheme, by definition $S' = (|S|, \mathcal{O}_{S,0})$ is an ordinary scheme, that is, for each point $x \in |S|$ there exists an open set $U \subset |S|$ such that $(U, \mathcal{O}_{S,0}|_U) = \underline{\operatorname{Spec}}\, A_0$ for an ordinary commutative algebra A_0.

By definition of superscheme, $\mathcal{O}_{S,1}|_U$ is a quasi-coherent sheaf of $\mathcal{O}_{S,0}|_U$-modules, i.e., of $\mathcal{O}_{A_0}$-modules. Hence there exists an A_0-module A_1 such that $\mathcal{O}_{S,1}|_U(U) = A_1$ (see Chapter 2, Section 2.5). So if we look at global sections we have

$$\mathcal{O}_S|_U = \mathcal{O}_{S,0}|_U \oplus \mathcal{O}_{S,1}|_U = \mathcal{O}_{A_0} \oplus \mathcal{O}_{A_1} = \mathcal{O}_A,$$

where $A := A_0 \oplus A_1$. Since $\mathcal{O}_S$ is a sheaf of superalgebras, A is also a superalgebra and one can readily check that $(U, \mathcal{O}_S|_U) = \underline{\operatorname{Spec}}\, A$. □

Given a superscheme $X = (|X|, \mathcal{O}_X)$, let $\mathcal{O}_X^r$ denote the sheaf of algebras

$$\mathcal{O}_X^r(U) = (\mathcal{O}_X/J_X)(U),$$

where J_X is the ideal sheaf $U \mapsto J_{\mathcal{O}_X(U)}$ with $J_{\mathcal{O}_X(U)}$ the ideal generated by the odd nilpotents in $\mathcal{O}_X(U)$.

We will call $X^r = (|X|, \mathcal{O}_X^r)$ the *reduced space* associated to the superspace $X = (|X|, \mathcal{O}_X)$. This is a locally ringed space in the classical sense. The scheme $(|X|, \mathcal{O}_X^r)$ is called the *reduced scheme* associated to X. Notice that the reduced scheme associated to a given superscheme may not be reduced, i.e., $\mathcal{O}_X^r(U)$, U open in $|X|$, can contain nilpotents. This is because $\mathcal{O}_X^r(U)$ is obtained by taking the quotient of $\mathcal{O}_X(U)$ by the ideal generated only by the *odd* nilpotents.

In Section 2.5 of Chapter 2 we have discussed quasi-coherent sheaves in the ordinary setting and we examined the equivalence of categories between the category of quasi-coherent sheaves of $\mathcal{O}_A$-modules over $\underline{\mathrm{Spec}}\, A$ and the category of A-modules for an ordinary commutative algebra A. We want to generalize this picture to the super setting.

Let A be a commutative superalgebra and let M be an A-module. We can regard M as an A_0-module and construct the sheaf $\widetilde{M}$ on $\underline{\mathrm{Spec}}\, A_0$ as in Section 2.5 of Chapter 2. However, since M is a module for the superalgebra A, we have that the sheaf $\widetilde{M}$ inherits naturally an $\mathcal{O}_A$-module structure, i.e., $\widetilde{M}(U)$ has an $\mathcal{O}_A(U)$-module structure for all $U \subset \mathrm{Spec}\, A_0$ and such a structure is compatible with restriction morphisms. We summarize all the properties of the sheaf $\widetilde{M}$ in a theorem, which is the supergeometric counterpart of Propositions 2.5.1 and 2.5.4. The proof is a direct check, very similar to the classical setting, and can be found for example in [43], Ch. II; we leave it to the reader.

Theorem 10.1.4. *Let M be an A-module for a superalgebra A, and let $\widetilde{M}$ be as above. Then:*

(1) *$\widetilde{M}$ has a natural structure of $\mathcal{O}_A$-module.*

(2) *$(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\, A_0$, i.e., the stalk at any prime $\mathfrak{p}$ of the sheaf $\widetilde{M}$ coincides with the localization of the A_0-module M at $\mathfrak{p}$.*

(3) *$(\widetilde{M})(\mathrm{Spec}\, A_0) = M$, i.e., the global sections of the sheaf coincide with the A-module M.*

Again we have as in the classical setting an equivalence of categories:

Proposition 10.1.5. *The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_A$-modules. The inverse of this functor is the functor $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}\, A_0)$.*

As in the classical setting we can define the notion of closed subsuperschemes.

Definition 10.1.6. We say that $Y = (|Y|, \mathcal{O}_Y)$ is a *closed subscheme* of a superscheme $X = (|X|, \mathcal{O}_X)$ if:

(1) $|Y|$ is a closed subset of $|X|$ and Y is a superscheme.

(2) $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$ where $\mathcal{I}$ is a quasi-coherent sheaf of ideals in $\mathcal{O}_X$. In other words, for each $x \in |X|$ there exists an open affine set U, $x \in U$, such that $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Remark 10.1.7. If $X = \underline{\mathrm{Spec}}\, A$, the closed subschemes of X are in one-to-one correspondence with ideals in A as it happens in the ordinary case. Such a correspondence is realized in the following way:

$$\left\{ \begin{array}{c} \text{quasi-coherent sheaves} \\ \text{of ideals in } \mathcal{O}_A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{two-sided homogeneous} \\ \text{ideals in } A \end{array} \right\}$$

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathcal{I}(\operatorname{Spec} A_0) \\ \tilde{I} & \longleftarrow & I. \end{array}$$

Here $\tilde{I}$ is the quasi-coherent sheaf associated to the A_0-module I.

Examples 10.1.8. (1) *Affine superspace* $\mathbb{A}^{m|n}$. Consider the polynomial superalgebra $k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$ over an algebraically closed field k, where $x_1, \dots, x_m$ are even indeterminates and $\xi_1, \dots, \xi_n$ are odd indeterminates (see Chapter 1). We define $\underline{\operatorname{Spec}}\, k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$ to be the *affine superspace* of superdimension $m|n$ and denote it by $\mathbb{A}^{m|n}$.

The topological space underlying $\mathbb{A}^{m|n}$ is $\operatorname{Spec} k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]_0$ and consists of the even maximal ideals

$$(x_i - a_i, \xi_j \xi_k), \quad i = 1, \dots, m, \ j, k = 1, \dots, n,$$

and the even prime ideals

$$(p_1, \dots, p_r, \xi_j \xi_k), \quad j, k = 1, \dots, n,$$

where $(p_1, \dots, p_r)$ is a prime ideal in $k[x_1, \dots, x_m]$. In other words the prime ideals in $k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]_0$ are generated by the prime ideals in $k[x_1, \dots, x_m]$ and the even nilpotent ideal $(\xi_i \xi_j, i \le j)$.

At the prime ideal $\mathfrak{p} \in \operatorname{Spec} k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]_0$ the stalk of the structure sheaf of $\mathbb{A}^{m|n}$ is

$$\mathcal{O}_{\mathbb{A}^{m|n}, \mathfrak{p}} = \left\{ \tfrac{f}{g} \mid f \in \mathcal{O}_{\mathbb{A}^{m|n}}(\mathbb{A}^m), \ g \in \mathcal{O}_{\mathbb{A}^{m|n}}(\mathbb{A}^m)_0, \ g \notin \mathfrak{p} \right\}.$$

(2) *Supervariety over the sphere* S^2. Consider the polynomial superalgebra generated over an algebraically closed field k, $k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]$, and the ideal

$$I = (x_1^2 + x_2^2 + x_3^2 - 1, \ x_1 \cdot \xi_1 + x_2 \cdot \xi_2 + x_3 \cdot \xi_3).$$

Let $k[X] = k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]/I$ and $X = \underline{\operatorname{Spec}}\, k[X]$. Then X is a supervariety whose reduced variety X^r is the sphere S^2. A maximal ideal in $k[X]_0$ is given by

$$\mathbf{m} = (x_i - a_i, \ \xi_i \xi_j) \quad \text{with } i, j = 1, 2, 3, \ a_i \in k \text{ and } a_1^2 + a_2^2 + a_3^2 = 1.$$

The local ring of $k[X]_0$ at the maximal ideal $\mathbf{m}_0$ is the ring of fractions

$$(k[X]_0)_{\mathbf{m}} = \left\{ \tfrac{f}{g} \mid f, g \in k[X]_0, \ g \notin \mathbf{m} \right\}.$$

The stalk of the structure sheaf at $\mathbf{m}$ is the localization of $k[X]$ as a $k[X]_0$-module, that is,

$$k[X]_{\mathbf{m}} = \{\tfrac{m}{g} \mid m \in k[X],\ g \in k[X]_0,\ g \notin \mathbf{m}\}.$$

Notice that if $a_1 \neq 0$ (not all a_i are zero simultaneously), then x_1 is invertible in the localization and we have

$$\xi_1 = -\frac{1}{x_1}(x_2\xi_2 + x_3\xi_3),$$

so $\{\xi_2, \xi_3\}$ generate $k[X]_{\mathbf{m}}$ as an $\mathcal{O}_{k[X]_0}$-module.

X is an example of a closed subscheme of the affine superspace $\mathbb{A}^{3|3}$. Notice also that the reduced scheme corresponding to X is the unitary sphere $S^2 \subset \mathbb{A}^3$.

We are now going to see that the category of affine superschemes is equivalent to the category of superalgebras; in other words, the superscheme $X = \underline{\mathrm{Spec}}\, A$ and the superalgebra A contain the same information. This is a generalization of the equivalence between the category of affine schemes and the category of algebras (for a proof of the classical result see Chapter 2).

Let (aschemes) denote the category of affine superschemes. Let us define the functor $F\colon (\mathrm{salg})^{\mathrm{op}} \to (\mathrm{aschemes})$ on the objects as $F(A) = \underline{\mathrm{Spec}}\, A$. In order to define F on the morphisms, let $\phi\colon A \to B$ and $\phi_0 = \phi|_{A_0}$. We need to give a morphism $f = F(\phi)\colon \underline{\mathrm{Spec}}\, B \to \underline{\mathrm{Spec}}\, A$. On the topological space we have immediately $|f|\colon \mathrm{Spec}\, B_0 \to \mathrm{Spec}\, A_0$ defined as $|f|(\mathfrak{p}) = \phi_0^{-1}(\mathfrak{p})$. For the sheaf morphism, we need to give a family of morphisms,

$$f_U^*\colon \mathcal{O}_A(U) \to \mathcal{O}_B(|f|^{-1}(U)), \quad U \text{ open in } \mathrm{Spec}\, A_0,$$

commuting with restrictions. If $a \in \mathcal{O}_A(U)$, i.e., $a\colon U \to \coprod_{x \in U} \mathcal{O}_{A,x}(U)$ (see Definition 2.2.6), define $f_U^*(a)\colon \mathfrak{p} \mapsto \phi_{\mathfrak{p}}(a(|f|(\mathfrak{p})))$ and $\phi_{\mathfrak{p}}\colon A_{\phi_0^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Spec}\, B_0$. One can check that this determines a sheaf morphism $f^*\colon \mathcal{O}_A \to f_*\mathcal{O}_B$ and that such f^* is local.

We now claim that F gives an equivalence between the category of superalgebras and the category of affine superschemes. In order to show that F realizes an equivalence of categories, we need to prove that there is a one-to-one correspondence between objects and morphisms in the categories. We are going to do this by establishing an inverse functor G. On the objects the inverse functor is given by

$$G\colon (\mathrm{aschemes}) \to (\mathrm{salg}), \quad \underline{\mathrm{Spec}}\, A \mapsto \mathcal{O}_A(A_0) \cong A.$$

On the morphisms G is given as follows. Let $f\colon \underline{\mathrm{Spec}}\, B \to \underline{\mathrm{Spec}}\, A$ be a morphism. Since the global sections of the structure sheaves coincide with the rings B and A respectively, we obtain immediately a morphism from A to B, so we set

$$G(f) = f^*_{\mathrm{Spec}\, A_0}\colon \mathcal{O}_A(\mathrm{Spec}\, A_0) \cong A \to \mathcal{O}_B(\mathrm{Spec}\, B_0) \cong B.$$

We leave it to the reader to verify that F and G are functors and moreover that $F \cdot G \cong \mathrm{id}$ and $G \cdot F \cong \mathrm{id}$.

We have proven the following proposition.

Proposition 10.1.9. *There exists an equivalence of categories between the category of commutative superalgebras and the category of affine superschemes.*

When we restrict the functor F to the category of affine superalgebras, it gives an equivalence of categories between affine superalgebras and affine supervarieties.

We now would like to give an example of a not affine superscheme which is of particular importance: the projective superspace.

Example 10.1.10 (Projective superspace). Let $S = k[x_0, \dots, x_m, \xi_1, \dots, \xi_m]$. Then S is a $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ graded algebra and the two gradings are compatible. Define the topological space Proj S_0 as the set of $\mathbb{Z}$-homogeneous non-irrelevant primes in S_0 (i.e., those primes not containing $(x_0, \dots, x_m, \xi_i \xi_j)$), with the Zariski topology. Proj S_0 is covered by open affine sets U_i consisting of those primes not containing (x_i). That is to say that U_i consists of the homogeneous primes in $k[x_0, \dots, x_m, \xi_1, \dots, \xi_n][x_i^{-1}]$ which in turn can be identified with the primes in the ring of elements of degree zero in $k[x_0, \dots, x_m, \xi_1, \dots, \xi_n][x_i^{-1}]$. One can readily see that such ring is isomorphic to $k[u_0, \dots, \hat{u}_i, \dots, u_m, \xi_1, \dots, \xi_n]$. Hence, as in the classical setting (see Example 2.3.4), we have

$$U_i = \operatorname{Spec} k[u_0, \dots, \hat{u}_i, \dots, u_m, \xi_1, \dots, \xi_n]_0, \quad i = 0, \dots, m.$$

We can define the sheaves

$$\mathcal{O}_{U_i} = \mathcal{O}_{k[u_0, \dots, \hat{u}_i, \dots, u_m, \xi_1, \dots, \xi_n]}$$

so that $(U_i, \mathcal{O}_{U_i})$ are affine superschemes corresponding to these open affine subsets. One can check that the conditions of Proposition 2.2.12 are satisfied, hence these sheaves glue together to give a sheaf $\mathcal{O}_S$ on all the topological space Proj S. So we define the *projective superspace* $\mathbb{P}^{m|n}$, denoted also by $\underline{\text{Proj}}\, S$, as the superscheme $(\text{Proj}\, S_0, \mathcal{O}_S)$.

The same construction can be easily repeated for a generic $\mathbb{Z}$-graded superalgebra.

10.2 The functor of points

As for supermanifolds, we employ the functor of points approach from algebraic geometry to better handle the nilpotent elements and to recover the geometric intuition. It is very important to realize that in ordinary algebraic geometry one could introduce this subject at an elementary level, avoiding altogether this level of abstraction and deal only with algebraic varieties described by prime ideals in a polynomial algebra and their regular functions over an algebraically closed field. On the contrary here, even when the superscheme has a reduced irreducible underlying ordinary scheme (that is, it corresponds to an algebraic variety), this machinery is unavoidable and often the functor of points and the representability criterion are the only means we have to handle these objects in the algebraic setting, even when the field is algebraically closed.

Definition 10.2.1. For a superscheme X, the *functor of points* of X is the representable functor

$$h_X : (\text{sschemes})^{\text{op}} \to (\text{sets}), \quad h_X(Y) = \text{Hom}(Y, X),$$

defined as usual on the morphisms as $h_X(\phi)(\psi) = \psi \circ \phi$, $\phi : Y \to Z$. The elements in $h_X(Y)$ are called the *Y-points* of the superscheme X.

In the previous chapters, we have mostly used the same notation to denote both a supergeometric object, say a supermanifold, and its functor of points. In this chapter, however, we want to make a distinction since we will also deal with non-representable functors and with representability issues.

As in the ordinary setting, the functor of points of a superscheme is determined by looking at its restriction to the category of affine superschemes. This result is obtained in the same way as the corresponding classical one. We briefly sketch the proof and describe the ideas involved (for more see [29], Ch. VI).

Definition 10.2.2. Let T be a superscheme. We say that $\{T_i\}$ is an *open cover* of T, if each T_i is an open subscheme of T and the T_i's cover T, that is $\bigcup_i |T_i| = |T|$. We further say that $\{T_i\}$ is an *open affine cover* of T, if it is an open cover and each T_i is affine, that is to say $T_i = \underline{\text{Spec}}\, A_i$ for superalgebras A_i.

Lemma 10.2.3. *Let X and T be superschemes and let $\{T_i\}$ be an affine open cover of T. Consider a family of morphisms $(\phi_i)_{i \in I}$, $\phi_i \in h_X(T_i)$, such that $h_X(\iota_{ij})(\phi_i) = h_X(\iota_{ji})(\phi_j)$, where $\iota_{ij} : T_i \cap T_j \hookrightarrow T_i$ and $\iota_{ji} : T_i \cap T_j \hookrightarrow T_j$. Then there exists a unique $\phi \in h_X(T)$ such that $h_X(\iota_i)\phi = \phi_i$, where $\iota_i : T_i \hookrightarrow T$.*

Notation. In algebraic geometry it is customary to summarize the statement of the lemma by saying that the following sequence is exact:

$$h_X(T) \to \prod h_X(T_i) \underset{h_X(\iota_{ji})}{\overset{h_X(\iota_{ij})}{\rightrightarrows}} \prod h_X(T_i \cap T_j).$$

Proof. We have to look for a morphism $\phi : T \to X$, which amounts to finding a pair consisting of a continuous map $|\phi| : |T| \to |X|$ and a sheaf morphism $\phi^* : \mathcal{O}_X \to \phi_* \mathcal{O}_T$. The hypothesis guarantees that since the continuous maps $|\phi_i| : |T_i| \to |X|$ agree on the intersections $|T_i| \cap |T_j|$, there exists a unique continuous map $|\phi| : |T| \to |X|$, which is $|\phi_i|$ when restricted to the $|T_i|$. For the sheaf morphism the reasoning is similar. Consider the commutative diagram

$$\begin{array}{ccc} T_i \cap T_j & \overset{\iota_{ij}}{\hookrightarrow} & T_i \\ & \searrow^{\phi_{ij}} & \downarrow \phi_i \\ & & X. \end{array}$$

Clearly there is a corresponding commutative diagram, with the arrows reversed, involving the sheaf morphisms (for clarity we remove the $|\cdot|$ that indicates the continuous maps between the topological spaces underlying our superschemes):

$$\begin{array}{ccc}
\mathcal{O}_{T_i}|_{T_i \cap T_j}(\phi_i^{-1}(U)) = \mathcal{O}_{T_i \cap T_j}(\phi_{ij}^{-1}(U)) & \longleftarrow & \mathcal{O}_{T_i}(\phi_i^{-1}(U)) \\
& \nwarrow & \uparrow \\
& & \mathcal{O}_X(U).
\end{array}$$

We want to define $\phi^* \colon \mathcal{O}_X \to \phi_* \mathcal{O}_T$, in other words, we want to define for all open $U \subset |X|$ $\phi_U^* \colon \mathcal{O}_X(U) \to \mathcal{O}_T(\phi^{-1}(U))$.

Clearly the diagram detailed above, for $f \in \mathcal{O}_X(U)$, gives us a family $\{f_i\}$, $f_i \in \mathcal{O}_{T_i}(\phi_i^{-1}(U)) = \mathcal{O}_T|_{T_i}(\phi_i^{-1}(U)) = \mathcal{O}_T(\phi_i^{-1}(U))$. The commutativity of the above diagram ensures that the f_i's glue together to give $f' \in \mathcal{O}_T(\phi^{-1}(U))$, so that we can define $\phi_U^*(f) = f'$. We leave to the reader the routine checks that show that ϕ^* so defined has all the necessary properties.

Uniqueness is clear. □

Proposition 10.2.4. *Let X and T be superschemes and let $\phi \colon T \to X$ be a morphism. Then ϕ is determined by its restrictions to an open affine cover of T.*

Proof. Let $\{T_i\}$ be an open affine cover of T and let $\phi_i = \phi|_{T_i}$. As an exercise one can check that the ϕ_i satisfy the hypothesis of the previous lemma, hence we have that there exists a unique morphism $T \to X$, whose restriction to each T_i is ϕ_i. Such a morphism hence coincides with ϕ. □

Theorem 10.2.5. *Let X be a superscheme, h_X its functor of points. Then h_X is determined by its restriction to the category of affine superschemes.*

Proof. Assume that we know $h_X(Z)$ for all affine superschemes Z. Any morphism ϕ in $h_X(T)$ corresponds to a family of morphisms $\phi_i \in h_X(T_i)$ for an open cover of T. Hence it can be uniquely extended to a morphism in $h_X(T)$, necessarily ϕ. □

Since we have the contravariant equivalence of categories detailed in Proposition 10.1.9, we can view the restriction of h_X to the category of affine superschemes also as a functor:

$$h_X^a \colon (\text{salg}) \to (\text{sets}), \quad h_X^a(A) = \mathrm{Hom}_{(\text{sschemes})}(\underline{\mathrm{Spec}}\, A, X).$$

When the superscheme X is affine, i.e., $X = \underline{\mathrm{Spec}}\, R$, h_X^a is representable. In fact by Proposition 10.1.9,

$$h_X^a(A) = \mathrm{Hom}_{(\text{sschemes})}(\underline{\mathrm{Spec}}\, A, \underline{\mathrm{Spec}}\, R) = \mathrm{Hom}_{(\text{salg})}(R, A).$$

Observation 10.2.6. Since we have the equivalence of categories between affine superschemes and superalgebras, we can define an affine superscheme $X = \underline{\mathrm{Spec}}\, R$ equivalently as a representable functor

$$(\text{salg}) \to (\text{sets}), \quad A \mapsto \mathrm{Hom}_{(\text{salg})}(R, A).$$

With an abuse of notation we shall use h_X^a ($X = \underline{\mathrm{Spec}}\, R$) to denote this functor as well.

Remark 10.2.7. To simplify notation we drop the suffix a in h_X^a; the context will make clear whether we are considering h_X or its restriction to affine superschemes. Moreover, whenever we want the restriction of h_X to affine superschemes, we shall use the same symbol, $h_{\underline{\mathrm{Spec}}\, R}$, for the two functors

$$\begin{aligned}(\mathrm{salg}) \to (\mathrm{sets}), \quad & A \mapsto \mathrm{Hom}(R, A),\\ (\mathrm{aschemes}) \to (\mathrm{sets}), \quad & A \mapsto \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, \underline{\mathrm{Spec}}\, R).\end{aligned}$$

Observation 10.2.8. Let X^0 be an affine variety over an algebraically closed field k. Consider an affine supervariety X whose reduced part coincides with X^0. Then one can immediately check that the k-points of X correspond to the points of the affine variety X^0. For example, if $X = \mathbb{A}^{m|n}$ we have $X^0 = \mathbb{A}^m$ and, as we shall see in the next example, $h_X(k) = k^m = h_{X^0}(k)$.

Examples 10.2.9. (1) *Affine superspace revisited.* Let $A \in (\mathrm{salg})$ and let $V = V_0 \oplus V_1$ be a free supermodule (over k). Let (smod) denote the category of k-modules. Define

$$V(A) = (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1.$$

Notice that following a very common abuse of notation we are using the same letter V to denote both the super vector space V and its functor of points. The abuse will soon become worse when we shall also use V to denote the superscheme whose functor of points is V. In general this functor is not representable. However, if V is finite-dimensional, the functor is indeed representable and in fact we have

$$(A \otimes V)_0 \cong \mathrm{Hom}_{(\mathrm{smod})}(V^*, A) \cong \mathrm{Hom}_{(\mathrm{salg})}(\mathrm{Sym}(V^*), A),$$

where $\mathrm{Sym}(V^*)$ denotes the symmetric algebra over the dual space V^*. Recall that V^* is the set of linear maps $V \to k$ not necessarily preserving the parity, and $\mathrm{Sym}(V^*) = \mathrm{Sym}(V_0^*) \otimes \bigwedge V_1^*$, where $\bigwedge V_1^*$ denotes the exterior algebra over the ordinary space V_1.

Let us fix a basis for V and let $\dim V = p|q$. The functor V is represented by

$$k[V] = k[x_1, \dots, x_p, \xi_1, \dots, \xi_q],$$

where x_i and ξ_j are respectively even and odd indeterminates.

Hence the functor V is the functor of points of the affine supervariety $\mathbb{A}^{m|n}$ introduced in Example 10.1.8 (1).

We also want to remark that the functor D_V defined as

$$D_V(A) := \mathrm{Hom}_{(\mathrm{smod})}(V, A)$$

is representable for any V (not necessarily finite-dimensional), and it is represented by the superalgebra $\mathrm{Sym}(V)$. Clearly $V \cong D_V$ when V is finite-dimensional.

(2) *Supermatrices revisited*. Let $A \in (\text{salg})$. Define $\mathrm{M}_{m|n}(A)$ as the set of endomorphisms of the A-supermodule $A^{m|n}$. Choosing coordinates we can write

$$\mathrm{M}_{m|n}(A) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \right\},$$

where a, b are $m \times m$, $n \times n$ blocks of even elements in A and α, β are $m \times n$, $n \times m$ blocks of odd elements in A.

This is the functor of points of an affine supervariety represented by the commutative superalgebra: $k[\mathrm{M}(m|n)] = k[x_{ij}, \xi_{kl}]$ where x_{ij}'s and ξ_{kl}'s are respectively even and odd variables with $1 \leq i, j \leq m$ or $m+1 \leq i, j \leq m+n$, $1 \leq k \leq m$, $m+1 \leq l \leq m+n$ or $m+1 \leq k \leq m+n$, $1 \leq l \leq m$.

Notice that $\mathrm{M}_{m|n} \cong h_{\mathbb{A}^{m^2+n^2|2mn}}$.

Now consider the invertible matrices in $\mathrm{M}_{m|n}(A)$. This clearly gives us another functor

$$(\text{salg}) \to (\text{sets}), \quad A \mapsto \text{invertible matrices in } \mathrm{M}_{m|n}(A).$$

As we shall see in the next chapter, this is the functor of points of the superscheme $\mathrm{GL}_{m|n}$ introduced in Example 3.3.4, though this fact is far from being immediate.

This example raises a very natural question: how can we determine whether a functor $F: (\text{salg}) \to (\text{sets})$ is the functor of points of a superscheme? We can reformulate this question in an equivalent way: can we give a representability criterion for functors $(\text{sschemes}) \to (\text{sets})$? These questions have a positive answer in the classical setting and a comprehensive treatment can be found in [23], Ch. I, and in [29], Ch. VI. In the next section we are going to see how the classical argument can be replicated, with small changes, in the supergeometric environment.

10.3 A representability criterion

We now want to single out among all the functors $F: (\text{salg}) \to (\text{sets})$ those that are the functor of points of superschemes. As we shall presently see, they are characterized by two key properties. The first property is *locality*, which is the analogue of the gluing property in the sheaf definition. Local functors are in fact also-called *Zariski sheaves* in the literature. The word Zariski refers to the fact that essentially we are considering the Zariski topology on the spectrum of the rings.[1] The second property is more delicate and corresponds to the fact that these functors locally look like the functor of points of affine superschemes.

Let us recall some notation from Chapter 2, Section 2.3.

Let A be a superalgebra. Given $f \in A_0$, let A_f denote

$$A_f = A[f^{-1}] := \{a/f^n \mid a \in A\}.$$

[1]There are other relevant topologies for this theory, but we are unable to treat them in our text.

The sets $U_f = \mathrm{Spec}(A_f)_0 = \{\mathfrak{p} \in \mathrm{Spec}\, A_0 | (f) \not\subset \mathfrak{p}\}$ are the basic open sets in the topological space $X = \mathrm{Spec}\, A_0$ (see Chapter 3). In fact by definition the open sets in the Zariski topology of Spec A_0 are

$$U_I = \{\mathfrak{p} \in \mathrm{Spec}\, A_0 \mid I \not\subset \mathfrak{p}\}$$

for all the ideals I in A_0.

Let us start with the notion of *local functor.*

Definition 10.3.1. Let $F\colon (\mathrm{salg}) \to (\mathrm{sets})$ be a functor. Let $\{f_i\}_{i\in I}$ be a family of elements in A_0, $(f_i, i \in I) = A_0$ and let $\phi_i : A \to A_{f_i}$ and $\phi_{ij} : A_{f_i} \to A_{f_i f_j}$ be the natural morphism. We say that the functor F is *local* or that it is a *sheaf in the Zariski topology* if for any family $\{\alpha_i\}_{i\in I}$, $\alpha_i \in F(A_{f_i})$, such that $F(\phi_{ij})(\alpha_i) = F(\phi_{ji})(\alpha_j)$ for all i and j, there exists a unique $\alpha \in F(A)$ with $F(\phi_i)(\alpha) = \alpha_i$.

As we shall presently see, this is a far reaching generalization of the concept of sheaf on a topological space, as we have described it in Chapter 2.

Let F be a local functor. By Observation 10.2.6, we can regard it as a functor $F\colon (\mathrm{aschemes})^{\mathrm{op}} \to (\mathrm{sets})$, via the equivalence of categories described in Proposition 10.1.9. Let F_A denote its restriction to the affine open subschemes of $\underline{\mathrm{Spec}}\, A$. Since these subsets form a basis for the topology on Spec A, we are going to see that F_A is a $\mathcal{B}$-sheaf in the sense specified in Chapter 2, where we forget the scheme structure on the affine open subschemes of $\underline{\mathrm{Spec}}\, A$ and treat them just as open sets in the topological space Spec A, their morphisms being the inclusions.

Since F_A is a functor it is automatically a presheaf in the Zariski topology. If $U = \mathrm{Spec}\, \mathcal{O}(U)$ and $V = \mathrm{Spec}\, \mathcal{O}(V)$ are two open affine sets in the topological space Spec A, the very definition of F_A, as a functor from the category of the open affine sets in Spec A to (sets), tells us that F_A is a presheaf of sets. In fact, if $i : U \hookrightarrow V$ is the inclusion of two open sets, one sees immediately that $F_A(i)\colon F_A(V) \to F_A(U)$ satisfies the properties of the restriction morphism, customarily denoted as $\alpha|_U = F_A(i)(\alpha)$, for $\alpha \in F_A(V)$.

We are now going to see that the locality of F guarantees the gluing of any family of local sections which agree on the intersection of any two open sets of an open covering.

Take an open affine covering $\{U_{f_i}\}_{i\in I}$ of $\underline{\mathrm{Spec}}\, A$, that is, $\mathrm{Spec}\, A_0 = \bigcup \mathrm{Spec}\, A_{f_i,0}$, $U_{f_i} = \mathrm{Spec}\, A_{f_i,0}$. We can assume that for our problem any open covering to be of this kind without any loss of generality since in the topological space Spec A_0, $\{U_f = \underline{\mathrm{Spec}}\, A_f\}_{f\in A_0}$ is a base for the open sets. The condition $\mathrm{Spec}\, A_0 = \bigcup \mathrm{Spec}\, A_{f_i,0}$ is equivalent to saying that $(f_i, i \in I) = A_0$. The condition that F is local can hence be translated in this context in the following way. Given any family $\{\alpha_i\}_{i\in I}$ with $\alpha_i \in F_A(U_{f_i})$, such that $\alpha_i|_{U_{f_i}\cap U_{f_j}} = \alpha_j|_{U_{f_i}\cap U_{f_j}}$, there exists a unique $\alpha \in F_A(\mathrm{Spec}\, A_0)$ such that $\alpha|_{U_{f_i}} = \alpha_i$ (where, again, we denote by $\alpha_i|_{U_{f_i}\cap U_{f_j}}$ the image of α_i under $F_A(\iota_{ij})$, where $\iota_{ij} : U_{f_i} \cap U_{f_j} \hookrightarrow U_{f_i}$). This is precisely the condition to impose on the presheaf F_A in order to have a $\mathcal{B}$-sheaf (see Chapter 2).

The examples of local functors are many; the one that interests us most is the following.

Proposition 10.3.2. *The functor of points h_X of a superscheme X is local.*

Proof. We briefly sketch the proof since it is the same as in the ordinary case. Let the notation be as above. Consider a collection of maps $\alpha_i \in h_X(A_{f_i})$ which map to the same element in $h_X(A_{f_i f_j})$. Each α_i consists of a continuous map $|\alpha_i|\colon$ Spec $A_{f_i\,0} \to |X|$ and a family of $\alpha^*_{i,U}\colon \mathcal{O}_X(U) \to \mathcal{O}_{A_{f_i}}(|\alpha|^{-1}(U))$ respecting restrictions. The fact the $|\alpha_i|$ glue together is clear. The gluing of the α_i^*'s to give $\alpha\colon \underline{\text{Spec}}\, A \to X$ depends on the fact that $\mathcal{O}_X$ and $\mathcal{O}_{A_{f_i}}$ are sheaves. □

We now want to define the second ingredient for our representability criterion, namely the notion of open subfunctor of a functor $F\colon$ (salg) $\to$ (sets). If we assume F to be the functor of points of superscheme X, there is a sensible notion of an open subfunctor of $F = h_X$, namely, we could simply define it as the functor of points of an open superscheme $U \subset X$. However since we are precisely interested in a characterization of those F that come from superschemes, we have to carefully extend the notion of open subfunctor to subfunctors of functors which do not come necessarily as the functor of points of a superscheme.

Definition 10.3.3. Let U be a subfunctor of a functor $F\colon$ (salg) $\to$ (sets); this means that we have a natural transformation $U \to F$ such that $U(A) \to F(A)$ is injective for all A. We say that U *is an open subfunctor of* F if for all $A \in$ (salg), given any natural transformation $f: h_{\underline{\text{Spec}}\, A} \to F$, the subfunctor $f^{-1}(U)$ coincides with h_V for some open V in $\underline{\text{Spec}}\, A$, where

$$f^{-1}(U)(R) := f_R^{-1}(U(R)), \quad f_R\colon h_{\underline{\text{Spec}}} A(R) \to F(R).$$

We say U is an *open affine subfunctor* of F if it is open and representable, in other words $U \cong h_{\underline{\text{Spec}}} R$ for some superalgebra R.

Very naturally, an open subfunctor of the functor of points of a superscheme X is h_U where U is open in X, as we shall see in the next observation.

Observation 10.3.4. Let $X = (|X|, \mathcal{O}_X)$ be a superscheme and $U \subset X$ open affine in X. Then h_U is an open affine subfunctor of h_X.

By Yoneda's lemma $f: h_{\underline{\text{Spec}}\, A} \to h_X$ corresponds to a map $f'\colon \underline{\text{Spec}}\, A \to X$. Let $V = f'^{-1}(U)$ open in $\underline{\text{Spec}}\, A$. We claim

$$f_R^{-1}(h_U(R)) = h_V(R) \quad \text{for all } R \in \text{(salg)}.$$

Let $\phi \in h_{\underline{\text{Spec}}\, A}(R)$, then $f_R(\phi) = f' \cdot \phi \in h_X(R)$.

Hence if $f_R(\phi) \in h_U(R)$ immediately we have that ϕ factors out as:

$$\phi\colon \underline{\text{Spec}}\, R \to V = f^{-1}(U) \to \underline{\text{Spec}}\, A.$$

So $f_R(\phi) \in h_U(R)$ if and only if $\phi \in h_V(R)$. We leave to the reader all the routine checks.

We want to define the notion of an open cover of a functor.

Definition 10.3.5. Let $F\colon (\text{salg}) \to (\text{sets})$ be a functor. F is covered by the open subfunctors $(U_i)_{i\in I}$, if and only if for any affine superscheme $\underline{\text{Spec}}\, A$ and natural transformation $f: h_{\underline{\text{Spec}}\, A} \to F$ we have that the fibered product $h_{\underline{\text{Spec}}\, A} \times_F U_i \cong h_{V_i}$ and $(V_i)_{i\in I}$ is an open cover of $\underline{\text{Spec}}\, A$. (For the definition of fibered product see the Appendix B).

Notice that, by the very definition of open subfunctor, the functor $h_{\underline{\text{Spec}}\, A} \times_F U_i$ is always representable. In fact, it is equal to $f^{-1}(U_i)$ which is by definition the functor of points of an open and affine V_i in $\underline{\text{Spec}}\, A$. Moreover, as before, we have that if F is the functor of points of a superscheme X, i.e., $F = h_X$, the family of open subfunctors $(U_i)_{i\in I}$ covers F if and only if $U_i = h_{W_i}$ and the W_i cover X, that is $|X| = \bigcup |W_i|$.

Remark 10.3.6. Notice also that asking $F(A) = \bigcup U_i(A)$ for all superalgebras A is far too restrictive. This is already a phenomenon we observe at the ordinary level. For example let us consider $F = h_{\underline{\text{Spec}}\, \mathbb{Z}}$. F admits $\{U_1 = h_{\underline{\text{Spec}}\, \mathbb{Z}[1/2]}, U_2 = h_{\underline{\text{Spec}}\, \mathbb{Z}[1/3]}\}$ as open affine cover, in fact $\{\text{Spec}\, \mathbb{Z}[1/2], \text{Spec}\, \mathbb{Z}[1/3]\}$ cover $\text{Spec}\, \mathbb{Z}$. However $F(\mathbb{Z}) \neq U_1(\mathbb{Z}) \cup U_2(\mathbb{Z})$ since $F(\mathbb{Z}) = \text{id}$, while $U_1(\mathbb{Z}) = U_2(\mathbb{Z}) = \emptyset$.

We are ready to state the main result of this section that allows us to characterize, among all the functors from (salg) to (sets), those which are the functors of points of superschemes.

Theorem 10.3.7. *A functor*

$$F\colon (\text{salg}) \to (\text{sets})$$

is the functor of points of a superscheme X, i.e., $F = h_X$ if and only if

(1) *F is local,*

(2) *F admits a cover by affine open subfunctors.*

Proof. The proof of this result is similar to that in the ordinary case detailed in [23], Ch. I, §1, 4, 4.4, but given its importance we shall rewrite it. We first observe that if h_X is the functor of points of a superscheme, by Proposition 10.3.2 it is local and by Definition 10.3.5 it admits a cover by open affine subfunctors.

Let us now assume that F satisfies the properties (1) and (2) of Theorem 10.3.7. We need to construct a superscheme $X = (|X|, \mathcal{O}_X)$ such that $h_X \cong F$. The construction of the topological space $|X|$ is the same as in the ordinary case. Let us sketch it.

Let $\{h_{X_\alpha}\}_{\alpha\in A}$ be the affine open subfunctors that cover F.[2] Define $h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta}$, ($X_{\alpha\beta}$ will correspond to the intersection of the two open affine X_α and X_β in

[2] We are now assuming that the open affine subfunctors of F are equal and not just isomorphic to h_{X_α}. We leave to the reader the checks necessary to verify all the statements in the slightly more general setting required by the theorem.

the superscheme X, that we are now constructing). Notice also that $h_{X_\alpha} \times_F h_{X_\beta}$ is representable, as we remarked after Definition 10.3.5.

We have the commutative diagram

$$\begin{array}{ccc} h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta} & \xrightarrow{j_{\beta,\alpha}} & h_{X_\beta} \\ \Big\downarrow j_{\alpha,\beta} & & \Big\downarrow i_\beta \\ h_{X_\alpha} & \xrightarrow{i_\alpha} & F. \end{array}$$

As a set we define

$$|X| := \coprod_{\alpha \in A} |X_\alpha| / \sim,$$

where $\sim$ is the following relation: if $x_\alpha \in |X_\alpha|$, $x_\beta \in |X_\beta|$ then $x_\alpha \sim x_\beta$ if and only if there exists $x_{\alpha\beta} \in |X_{\alpha\beta}|$ such that $|j_{\alpha,\beta}|(x_{\alpha\beta}) = x_\alpha$ and $|j_{\beta,\alpha}|(x_{\alpha\beta}) = x_\beta$. Notice that we are (improperly) using the symbol $j_{\alpha,\beta}$ also for the superscheme morphism $j_{\alpha,\beta} \colon X_{\alpha,\beta} \to X_\alpha$. As one can check, this is an equivalence relation, $|X|$ is a topological space and $\pi_\alpha \colon |X_\alpha| \hookrightarrow |X|$ is an open continuous injective map.

We now need to define a sheaf of superalgebras $\mathcal{O}_X$. We shall use the sheaves in the open affine X_α's, "gluing" them appropriately.

Let U be open in $|X|$ and let $U_\alpha = \pi_\alpha^{-1}(U)$. Put

$$\mathcal{O}_X(U) := \left\{ (f_\alpha) \in \textstyle\coprod_{\alpha \in A} \mathcal{O}_{X_\alpha}(U_\alpha) \mid j_{\beta,\gamma}^*(f_\beta) = j_{\gamma,\beta}^*(f_\gamma) \text{ for all } \beta, \gamma \in A \right\}.$$

The condition $j_{\beta,\gamma}^*(f_\beta) = j_{\gamma,\beta}^*(f_\gamma)$ simply states that to be an element of $\mathcal{O}_X(U)$, the collection $\{f_\alpha\}$ must be such that f_β and f_γ agree on the intersection of X_β and X_γ for any β and γ. One can check that $\mathcal{O}_X$ is a sheaf of superalgebras.

Hence we have defined a superspace $X = (|X|, \mathcal{O}_X)$. X is a superscheme since, as a superspace, it admits a cover by open affine subschemes whose functor of points are the X_α's.

To finish the proof, we need to show that $h_X \cong F$. We are looking for a functorial bijection between $h_X(A) = \mathrm{Hom}_{(\mathrm{sschemes})}(\underline{\mathrm{Spec}}\, A, X)$ and $F(A)$ for all $A \in (\mathrm{salg})$. It is here that we use the hypothesis of F being local.

We first construct a natural transformation $\rho_A \colon F(A) \to h_X(A)$.

Let $t \in F(A) = \mathrm{Hom}(h_{\underline{\mathrm{Spec}}\, A}, F)$, by Yoneda's lemma. Consider the diagram

$$\begin{array}{ccc} h_{T_\alpha} := h_{X_\alpha} \times_F h_{\underline{\mathrm{Spec}}\, A} & \longrightarrow & h_{\underline{\mathrm{Spec}}\, A} \\ \Big\downarrow t_\alpha & & \Big\downarrow t \\ h_{X_\alpha} & \xrightarrow{i_\alpha} & F. \end{array}$$

Since the h_{X_α} are an open affine cover of F, the $\{T_\alpha\}$ form an open affine cover of $\underline{\mathrm{Spec}}\, A$. Since by Yoneda's lemma $\mathrm{Hom}(h_{T_\alpha}, h_{X_\alpha}) \cong \mathrm{Hom}(T_\alpha, X_\alpha)$, we obtain a morphism $t_\alpha \colon T_\alpha \to X_\alpha \subset X$. One can check that the morphisms t_α glue together to give a morphism $t' \colon \underline{\mathrm{Spec}}\, A \to X$, hence $t' \in h_X(A)$. So we define $\rho_A(t) = t'$.

Next we construct another natural transformation $\sigma_A \colon h_X(A) \to F(A)$, which turns out to be the inverse of ρ.

Assume that $f \in h_X(A)$, i.e., $f \colon \underline{\mathrm{Spec}}\, A \to X$. Let $T_\alpha = f^{-1}(X_\alpha)$. By our hypothesis, the T_α's form a cover of $\underline{\mathrm{Spec}}\, A$ that we can assume to be affine. We immediately obtain morphisms $g_\alpha \colon T_\alpha \to X_\alpha$. By Yoneda's lemma, g_α corresponds to a natural transformation $g_\alpha \colon h_{T_\alpha} \to h_{X_\alpha}$. Since F is local, the morphisms $i_\alpha \circ g_\alpha \colon h_{T_\alpha} \to h_{X_\alpha} \to F$ glue together to give a morphism $g \colon h_{\underline{\mathrm{Spec}}\, A} \to F$, i.e., an element $g \in F(A)$. Define $\sigma_A(f) = g$.

One can directly check that ρ and σ are indeed natural transformations, moreover $\rho \circ \sigma$ and $\sigma \circ \rho$ are the identity so that ρ and σ are inverse to each other. Hence $F \cong h_X$. □

An important consequence of this theorem is the existence of fibered products in the category of superschemes. We start with a lemma for the affine case.

Lemma 10.3.8. *Let $X = \underline{\mathrm{Spec}}\, \mathcal{O}(X)$, $Y = \underline{\mathrm{Spec}}\, \mathcal{O}(Y)$, $Z = \underline{\mathrm{Spec}}\, \mathcal{O}(Z)$. Then the fibered product $X \times_Z Y$ is an affine superscheme and*

$$X \times_Z Y = \underline{\mathrm{Spec}}\, \mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y).$$

Proof. The proof is a simple exercise that makes use of Proposition 10.1.9 and the universal properties of the tensor product and the fibered product. □

Corollary 10.3.9. *Fibered products exist in the category of superschemes. The fibered product $X \times_Z Y$, for superschemes X, Y, Z,*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{\ f\ } & Z, \end{array}$$

is the superscheme whose functor of points is $h_X \times_{h_Z} h_Y$.

Proof. We sketch the argument, since it is the same as in the ordinary case, found for example in [23] I §1, 5.1. Let $F \colon (\text{salg}) \to (\text{sets})$ be the functor $F(A) = h_X(A) \times_{h_Z(A)} h_Y(A)$. We want to show that F is the functor of points of a superscheme. We shall use Theorem 10.3.7. The fact that F is local, comes from its very definition and we leave it to the reader as an exercise. It is based on the fact that h_X, h_Y and h_Z are all local.

We now want to show that it can be covered by open affine subfunctors. Let $\{Z_i\}$ be an affine open covering of Z. Define $X_i = X \times_Z Z_i = f^{-1}(Z_i)$ and $Y_i = Y \times_Z Z_i = g^{-1}(Z_i)$. Notice that we know that X_i and Y_i are superschemes since they are open subschemes of X and Y, respectively. Let $X_{i\alpha}$ and $Y_{j\beta}$ be open affine covers of X_i and Y_j respectively. A straightforward check shows that both $h_{X_{i\alpha}} \times_{h_Z} h_Y$ and $h_X \times_{h_Z} h_{Y_{j\beta}}$ are open subfunctors of F. Moreover, since

$$h_{X_{i\alpha}}(A) \times_{h_{Z_i}(A)} h_{Y_{j\beta}}(A) = (h_{X_{i\alpha}}(A) \times_{h_Z(A)} h_Y(A)) \cap (h_X(A) \times_{h_Z(A)} h_{Y_{j\beta}}(A)),$$

also $h_{X_{i\alpha}} \times_{h_{Z_i}} h_{Y_{j\beta}}$ is an open subfunctor of F. Hence $\{h_{X_{i\alpha}} \times_{h_{Z_i}} h_{Y_{j\beta}}\}$ form an open covering of F by open subfunctors. These functors are affine since if $X_{i\alpha} = \underline{\mathrm{Spec}}\, R_{i\alpha}$, $Y_{j\beta} = \underline{\mathrm{Spec}}\, S_{j\beta}$ and $Z_i = \underline{\mathrm{Spec}}\, T_i$, $h_{X_{i\alpha}} \times_{h_{Z_i}} h_{Y_{j\beta}} = h_{\underline{\mathrm{Spec}}\, R_{i\alpha} \otimes_{T_i} S_{j\beta}}$

Hence by the representability criterion, Theorem 10.3.7, F is the functor of points of a superscheme, which is $X \times_Z Y$, as one can readily check. □

We end this section with some remarks on an equivalent approach to the problem discussed in Theorem 10.3.7.

As we have seen in the previous section, the functor of points of a superscheme X can be also equivalently defined as the functor $h_X : (\text{sschemes})^{\text{op}} \to (\text{sets})$, $h_X(T) = \mathrm{Hom}(T, X)$. Hence the question whether a generic functor $F : (\text{sschemes})^{\text{op}} \to (\text{sets})$ is (isomorphic to) the functor of points of a superscheme, can be reformulated by asking the conditions for F to be representable.

In such a setting, we can give an equivalent definition for F to be local, formulated as a proposition, whose proof is a straightforward check, based on the results of the previous section.

Let $F : (\text{sschemes})^{\text{op}} \to (\text{sets})$ be a functor. Since the category of affine superschemes is equivalent to the category of commutative superalgebras, we can regard F restricted to the category of affine superschemes as a functor from the category of superalgebras to the category of sets. We shall call such functor F^a.

Proposition 10.3.10. *Let $F : (\text{sschemes})^{\text{op}} \to (\text{sets})$ be a functor. Then F^a is local if and only if the following condition is satisfied. Let X be a superscheme, X_i an open cover of X. Assume that there is a family of elements $f_i \in F(X_i)$ such that $f_{ij} = f_{ji}$ for all i, j, where f_{ij} is the image of f_i under the natural map $F(X_i) \to F(X_i \cap X_j)$ induced by the inclusion $X_i \cap X_j \subset X_i$. Then there exists a unique $f \in F(X)$ mapping to each f_i under the natural map $F(X) \to F(X_i)$.*

The analogue of the notion of an open affine cover of such F (see Definition 10.3.5) makes sense as it is in this setting, hence we are ready for the Representability Theorem.

Theorem 10.3.11 (Representability criterion for superschemes). *A functor*

$$F : (\text{sschemes})^{\text{op}} \to (\text{sets})$$

is representable if and only if F is local, and F is covered by open affine subfunctors.

The proof is very similar to the proof of Theorem 10.3.7 and actually we have seen it in the context of differentiable supermanifolds in Theorem 9.4.3. Since the checks involved are very formal, that proof holds practically with no changes here and for this reason we leave it to the reader as an exercise.

10.4 The Grassmannian superscheme

In this section we want to discuss the Grassmannian superscheme corresponding to the $r|s$-dimensional superspaces inside a super vector space of dimension $m|n$, $r < m$, $s < n$. We will show that it is a superscheme using the representability criterion Theorem 10.3.7. This is a particularly important example since it is the first non-trivial example of a non-affine superscheme.

Consider the functor $\mathrm{Gr}\colon (\mathrm{salg}) \to (\mathrm{sets})$, where for any superalgebra A, $\mathrm{Gr}(A)$ is the set of projective A-submodules of rank $r|s$ of $A^{m|n}$ (for the definition of the rank of a projective A-module see Appendix B.3). When A is a local superalgebra, i.e., it contains a unique maximal ideal, one sees immediately from the definitions, that $\mathrm{Gr}(A)$ consists of the free A-submodules of dimension $r|s$ of the free module $A^{m|n}$. For example if A is a field, say $A = k$, $\mathrm{Gr}(k)$ are the super vector subspaces of dimension $r|s$ inside $k^{m|n}$, thus recovering the more familiar notion of Grassmannian, as the set of subspaces of a given dimension inside a vector space.

Equivalently, $\mathrm{Gr}(A)$ can also be defined as the set $\mathrm{Gr}(A) = \{\alpha\colon A^{m|n} \to L\}$, where α is a surjective morphism, L is a projective A-module of rank $r|s$, modulo the following equivalence relation: $\alpha \equiv \alpha'$ if and only if they have the same kernel. Notice that in this definition L also varies.

We need also to specify Gr on morphisms $\psi\colon A \to B$.

Given a morphism $\psi\colon A \to B$ of superalgebras, as usual we can provide B with the structure of an A-module in a natural way by setting

$$a \cdot b = \psi(a)b, \quad a \in A,\ b \in B.$$

Also, given an A-module L, we can construct the B-module $L \otimes_A B$. So given ψ and the element of $\mathrm{Gr}(A)$, $f\colon A^{m|n} \to L$, we can define an element of $\mathrm{Gr}(B)$ as follows:

$$\mathrm{Gr}(\psi)(f)\colon B^{m|n} = A^{m|n} \otimes_A B \to L \otimes_A B$$

where $L \otimes_A B$ is clearly a projective B-module of rank $r|s$. We want to show that Gr is the functor of points of a superscheme.

We will start by showing it admits a cover of open affine subfunctors. Consider the multi-index $I = (i_1, \dots, i_r|\mu_1, \dots, \mu_s)$ and the map $\phi_I\colon A^{r|s} \to A^{m|n}$, where $\phi_I(x_1, \dots, x_r|\xi_1, \dots, \xi_s)$ with $1 \le i_1 < \cdots < i_r \le m$, $1 \le \mu_1 < \cdots < \mu_s \le n$ is the $m|n$-uple with $x_1, \dots, x_r$ occupying the positions $i_1, \dots, i_r$ and $\xi_1, \dots, \xi_s$ occupying the positions $\mu_1, \dots, \mu_s$ and all the other positions are occupied by zero. For example, let $m = n = 2$ and $r = s = 1$. Then $\phi_{1|2}(x, \xi) = (x, 0|0, \xi)$.

Now define the subfunctors v_I of Gr as follows. The $v_I(A)$ are the morphisms $\alpha\colon A^{m|n} \to L$, i.e., the elements in $\mathrm{Gr}(A)$ such that $\alpha \circ \phi_I$ is surjective. Notice that if $\alpha \circ \phi\colon A^{r|s} \to A^{m|n} \to L$ is surjective, since $A^{r|s}$ and L are projective modules of the same rank, $\alpha \circ \phi$ is an *isomorphism*, hence L is free. This is crucial for the representability of v_I and ultimately of Gr.

The functor v_I is in fact representable: $v_I \cong h_{\mathrm{M}_{(m|n)\times(m-r|n-s)}}$, that is, v_I is isomorphic to the functor of points of $((m|n) \times (m - r|n - s))$-matrices. In fact

$v_I(A)$ consists of morphisms $\alpha\colon A^{m|n} \to L \cong A^{r|s}$, where we specify the images of $r|s$ elements in the canonical basis of $A^{m|n}$. We leave to the reader the easy checks involved.

We want now to show that the v_I are open affine subfunctors of Gr. The condition that v_I is an open subfunctor is equivalent to asking that $f^{-1}(v_I)$ be open for any morphism $f\colon \underline{\text{Spec}}\,A \to \text{Gr}$. So we fix one of such morphisms f.

By Yoneda's lemma, a map $f\colon \underline{\text{Spec}}\,A \to \text{Gr}$ corresponds to a point f in $\text{Gr}(A)$. We need to find an open subscheme V_I in $\underline{\text{Spec}}\,A$ such that

$$h_{V_I}(B) = f^{-1}(v_I(B)) = \{\psi\colon A \to B \mid f_B(\psi) \in v_I(B)\} \subset h_{\underline{\text{Spec}}\,A}(B)$$

for all $B \in (\text{salg})$, where $f_B\colon h_{\underline{\text{Spec}}\,A}(B) \to \text{Gr}(B)$. Notice that $f_B(\psi) = f \circ \psi^{\#}$, with $\psi^{\#}\colon \underline{\text{Spec}}\,B \to \underline{\text{Spec}}\,A$ is the morphism of affine schemes induced by ψ (we prefer $\psi^{\#}$ instead of the more cumbersome notation $\underline{\text{Spec}}\,\psi$).

To show that V_I is open, we fix the canonical basis in $A^{r|s}$ and we define

$$Z_{A,I}(f) := f \cdot \phi_I \in \text{Hom}(A^{r|s}, L).$$

We want to determine which conditions must be satisfied to have $f \in v_I(A)$. Since v_I is local (it is representable), we can assume without loss of generality that A is local (see also Appendix B), so that $Z_{A,I}(f) \in \text{M}_{r|s}(A)$. In this case we have that $f \in v_I(A)$ if and only if $Z_{A,I}(f)$ is invertible. By Chapter 1, Section 1.5, we know that this is equivalent to having the determinants, say d_1 and d_2, of the diagonal blocks of $Z_{A,I}(f)$, invertible. This is an open affine condition, but let us look at it in more detail.

We have $f \in v_I(A)$ if and only if d_1 and d_2 are invertible in A. Let now $\psi \in h_{\underline{\text{Spec}}\,A}(B)$, i.e., $\psi\colon A \to B$. We have $\psi \in f^{-1}(v_I(B))$ if and only if ψ factors via $A[d_1^{-1}, d_2^{-2}]$, in other words $\psi\colon A \to A[d_1^{-1}, d_2^{-1}] \to B$. So $\psi \in h_{\underline{\text{Spec}}\,A[d_1^{-1},d_2^{-2}]}(B)$.

This shows that the v_I's are open and affine. It remains to show that these subfunctors cover Gr.

Given $f \in \text{Gr}(A)$, that is a function from $h_{\underline{\text{Spec}}\,A} \to \text{Gr}$, we have that since f is surjective, there exists at least an index I so that $Z_{A,I}(f)$ is invertible, hence $f \in v_I(A)$ for this I. The above argument shows also that we obtain a cover of any $\underline{\text{Spec}}\,A$ by taking $v_I \times_{\text{Gr}} h_{\underline{\text{Spec}}\,A}$.

Next we want to show that Gr is local. We shall do this by identifying $\text{Gr}(A)$ with locally free coherent sheaves of rank $r|s$. By Appendix B.3, we have that any projective module is locally free and moreover, by Theorem 10.1.4, one obtains the functorial correspondence between (locally free) coherent sheaves and finitely generated (projective) modules in the super-setting. Hence we have:

$$\text{Gr}(A) \cong \{\mathcal{F} \subset \mathcal{O}_A^{m|n} \mid \mathcal{F} \text{ is a subsheaf of locally constant rank } r|s\},$$

where $\mathcal{O}_A^{m|n} = \mathcal{O}_A \otimes k^{m|n}$.

By its very definition this functor is local. This is perhaps best seen by using the correspondence detailed in Proposition 10.3.10:

$$\mathrm{Gr}(T) \cong \{\mathcal{F} \subset \mathcal{O}_T^{m|n} \mid \mathcal{F} \text{ is a subsheaf of locally constant rank } r|s\}.$$

In other words, we are considering the functor Gr extended to the category of superschemes: $\mathrm{Gr}\colon (\mathrm{sschemes})^{\mathrm{op}} \to (\mathrm{sets})$ (by a common abuse of notation we use the same symbol to denote it). If $\{T_i\}$ is an open cover of the superscheme T, assume that we have a family of subsheaves $\mathcal{F}_i \in \mathrm{Gr}(T_i)$, $\mathcal{F}_i \subset \mathcal{O}_{T_i}^{m|n} = \mathcal{O}_T^{m|n}|_{T_i}$, with $\mathcal{F}_i|_{T_i \cap T_j} = \mathcal{F}_j|_{T_i \cap T_j}$. Then it is clear the $\mathcal{F}_i$'s glue together to give a subsheaf $\mathcal{F} \subset \mathcal{O}_T^{m|n}$ on T, still locally constant of rank $r|s$, hence $\mathcal{F} \in \mathrm{Gr}(T)$.

We have shown that Gr is the functor of points of a superscheme that we will call the *Grassmannian superscheme* (or *superGrassmannian* for short) of $r|s$ subspaces in an $m|n$-dimensional space.

10.5 Projective supergeometry

The aim of this section is to give a brief introduction to projective supergeometry and to show that the Grassmannian superscheme described in the previous section is not projective, that is, it is not a subscheme of the projective superspace. This section is independent from the rest of our work and can be skipped in a first reading. Also: in this section we assume that the reader is comfortable with some basic algebraic geometry constructions, such as line bundles and cohomology groups. Such notions will not appear elsewhere in our work.

Let S be a $\mathbb{Z}$-graded superalgebra, with the $\mathbb{Z}$-grading compatible with the super grading. Denote by S^n the subalgebra of elements with degree n. Assume further that S is generated by S^1, the subalgebra the elements of degree 1. S is a $\mathbb{Z}$-graded S-module in a natural way. We can define, very much in the same way as the ordinary setting, the *twisted module* $S(n)$ as

$$S(n)^d = S^{n+d}.$$

Clearly also $S(n)$ is an S-module (recall that upper indices refer to the $\mathbb{Z}$-grading while lower indices to the $\mathbb{Z}_2$-grading).

Let M be a graded S-module. Then M is also an S_0-module, hence we can follow the classical recipe (see Chapter 2, Section 2.5) and build the sheaf on Proj S_0, $\widetilde{M}$. One can check right away that $\widetilde{M}(U)$ is an $\mathcal{O}_S(U)$-module for all open sets U in Proj S_0. We summarize the properties of $\widetilde{M}$ in the following theorem.

Theorem 10.5.1. *Let M be an S-module, for a $\mathbb{Z}$-graded superalgebra S. Then:*

(1) *$\widetilde{M}$ has a natural structure of $\mathcal{O}_S$-module.*

(2) $(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ *for all homogeneous prime ideals* $\mathfrak{p} \in \operatorname{Proj} S_0$, *i.e., the stalk at any prime* $\mathfrak{p}$ *of the sheaf* $\widetilde{M}$ *coincides with the localization of the* S_0*-module* M *at* $\mathfrak{p}$.

(3) *For all homogeneous* $f \in S^n$, $\widetilde{M}|_{U_f} = \widetilde{M_{(f)}}$, *where* U_f *is the basic open set in* $\operatorname{Proj} S_0$ *corresponding to* f, $M_{(f)}$ *are the elements of degree zero in the localization* M_f *(this is commonly called* ***projective localization****).*

Proof. We leave the proof to the reader as an exercise: it is done precisely as in the ordinary setting that is found in [43], Ch. II, Section 5. □

At this point one could define *supercoherent sheaves* as sheaves modeled after $\widetilde{M}$, in analogy to what we have done in Capter 2, but we shall not take this direction.

We now turn to the generalization of the *Serre's twisting sheaf* in order to characterize the morphisms from a superscheme to the projective superspace.

Definition 10.5.2. We define $\mathcal{O}_S(n)$ as the sheaf $\widetilde{S(n)}$. Note that $\mathcal{O}_S(1)$ is called Serre's *twisting sheaf.*

$\mathcal{O}_S(n)$ is a locally free $\mathcal{O}_S$-module of rank 1. The proof is essentially the same as the classical one (see [43], Ch. II, Section 5).

Definition 10.5.3. Let X be a superscheme. A *super line bundle* on X is a locally free $\mathcal{O}_X$-module of rank 1. Moreover we say that a super line bundle $\mathcal{L}$ is *generated by the global sections* $s_1, \dots, s_m, \sigma_1, \dots, \sigma_n$ if the images of such global sections in the stalk $\mathcal{L}_p$ generate the stalk for all $p \in |X|$.

For example, for $S = k[x_0, \dots, x_m, \xi_1, \dots, \xi_n]$ consider the line bundle $\mathcal{O}(1) := \mathcal{O}_S(1)$ on $\mathbb{P}^{m|n} = \underline{\operatorname{Proj}} S$. As one can readily check, this super line bundle is generated by the global sections $x_0, \dots, x_m, \xi_1, \dots, \xi_n$.

The next proposition relates the morphisms of a superscheme X into the projective superspace $\mathbb{P}^{m|n}$, with the set of global sections of super line bundles.

Proposition 10.5.4. *Let* X *be a superscheme.*

(1) *For any morphism* $\phi\colon X \to \mathbb{P}^{m|n}$, $\phi^*(\mathcal{O}(1))$ *is a super line bundle on* X *and is generated by the global sections* $\phi^*(x_i)$, $\phi^*(\xi_j)$, $i = 1, \dots, m$, $j = 1, \dots, n$.

(2) *Vice versa, if* $\mathcal{L}$ *is a super line bundle on a superscheme* X *and if* $s_0, \dots, s_m$, $\sigma_1, \dots, \sigma_n$ *are global sections generating* $\mathcal{L}$, *then there exists a unique morphism* $\phi\colon X \to \mathbb{P}^{m|n}$ *such that* $\mathcal{L} = \phi^*(\mathcal{O}(1))$ *and* $s_i = \phi^*(x_i)$, $\sigma_j = \phi^*(\xi_j)$, $i = 0, \dots, m$, $j = 1, \dots, n$.

Proof. This is an exercise, once one follows the proof for the ordinary setting; see [43], Ch. II, Section 7. □

For a generic sheaf of rings $\mathcal{F}$, let $\mathcal{F}^*$ denote the sheaf of invertible sections, i.e., $\mathcal{F}^*(U)$ are the invertible sections in $\mathcal{F}(U)$. The first cohomology group $H^1(|X_0|, \mathcal{O}^*_{X_0})$ for an ordinary scheme X_0 classifies the equivalence classes of line bundles on X_0 ([43], Ch. III, Section 4). The next proposition tells us that the same happens also for super line bundles. We leave the proof of this result to the reader since it is a precise replica of the classical one.

Proposition 10.5.5. *Let $X = (|X|, \mathcal{O}_X)$ be a superscheme. There is a one-to-one correspondence between the equivalence classes of super line bundles on $|X|$ and the elements in the cohomology group $H^1(|X|, \mathcal{O}_X^*)$.*

We now turn to the main goal of this section: we want to show that the Grassmannian superscheme cannot in general be embedded into any projective superspace. This statement appears for the analytic category in [56]; the reasoning here is the same, we include it for completeness. We start with some general considerations on super line bundles on superschemes.

Let X be a superscheme and let X_0 be its reduced scheme, i.e., $X = (|X|, \mathcal{O}_X)$, $X_0 = (|X|, \mathcal{O}_{X_0}) = (|X|, \mathcal{O}_X/\mathcal{I})$ where $\mathcal{I}$ is the ideal sheaf generated by the odd nilpotents.

Consider the exact sequence of sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0.$$

We can construct the usual long exact sequence:

$$\begin{aligned} 0 \to H^0(|X|, \mathcal{I}) \to H^0(|X|, \mathcal{O}_X^*) \to H^0(|X|, \mathcal{O}_{X_0}^*) \to H^1(|X|, \mathcal{I}) \\ \to H^1(|X|, \mathcal{O}_X^*) \xrightarrow{\phi} H^1(|X|, \mathcal{O}_{X_0}^*) \xrightarrow{\psi} H^2(|X|, \mathcal{I}) \to \cdots . \end{aligned}$$

As we remarked previously, $H^1(|X|, \mathcal{O}_X^*)$ is in one-to-one correspondence with the super line bundles on $|X|$, while $H^1(|X|, \mathcal{O}_{X_0}^*)$ is in one-to-one correspondence with the line bundles on $|X_0| = |X|$. The map ϕ tells us which super line bundles on $|X|$ restrict properly to line bundles on $|X_0|$. We are going to show that, for a specific Grassmannian superscheme X, the super line bundles in $H^1(|X|, \mathcal{O}_X^*)$ restrict via ϕ only to line bundles in $H^1(|X|, \mathcal{O}_{X_0}^*)$ which have no global sections. Since the projective morphisms are given via global sections of super line bundles, this will show that X does not admit any projective embedding.

Let us take $X = \mathrm{Gr}(1|1, 2|2)$ so that $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with projective coordinates $[x_0, x_1]$ and $[y_0, y_1]$ for the first and second copy of $\mathbb{P}^1$. Then $|X| = |X_0|$ has the following open covering:

$$\begin{aligned} |U_{00}| &= \{([1, u_0], [1, v_0])\}, \quad |U_{10}| = \{([u_1, 1], [1, v_0])\}, \\ |U_{01}| &= \{([1, u_0], [v_1, 1])\}, \quad |U_{11}| = \{([u_1, 1], [v_1, 1])\}. \end{aligned}$$

Consider the functor of points of X on *local superalgebras* (that is, superalgebras which have only one maximal homogeneous ideal) since by the arguments in Appendix B.2, we know such superalgebras completely determine the functor of points.

X is the quotient of $\mathrm{GL}_{2|2}$ by the parabolic subgroup

$$P(A) = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix} \subset \mathrm{GL}_{2|2}(A),$$

which is the stabilizer of the element $\langle e_1, \epsilon_1 \rangle \in X(A)$. This leads to the usual identification of points in $X(A)$ with matrices in $\mathrm{GL}_{2|2}(A)/P(A)$.

The open covering of $|X| = \mathbb{P}^1 \times \mathbb{P}^1$ described above induces the following open covering of X by the open affine subfunctors U_{ij}:

$$U_{00}(A) = \left\{ \begin{pmatrix} 1 & * & 0 & * \\ u_0 & * & \eta & * \\ 0 & * & 1 & * \\ \nu & * & v_0 & * \end{pmatrix} \right\}, \quad U_{10}(A) = \left\{ \begin{pmatrix} u_1 & * & \zeta & * \\ 1 & * & 0 & * \\ 0 & * & 1 & * \\ \nu & * & v_0 & * \end{pmatrix} \right\},$$

$$U_{01}(A) = \left\{ \begin{pmatrix} 1 & * & 0 & * \\ u_0 & * & \eta & * \\ \xi & * & v_1 & * \\ 0 & * & 1 & * \end{pmatrix} \right\}, \quad U_{11}(A) = \left\{ \begin{pmatrix} u_1 & * & \zeta & * \\ 1 & * & 0 & * \\ \xi & * & v_1 & * \\ 0 & * & 1 & * \end{pmatrix} \right\}.$$

Assume that we have a line bundle on X and let us see how sections transform under coordinate changes. Generic sections on $|U_{00}|$ and $|U_{11}|$ respectively are given by

$$\begin{aligned} U_{00} &\to U_{00} \times \mathbb{C}, \quad (u_0, v_0, \nu, \eta) \mapsto ((u_0, v_0, \nu, \eta), s(u_0, v_0, \nu, \eta)), \\ U_{11} &\to U_{11} \times \mathbb{C}, \quad (u_1, v_1, \zeta, \xi) \mapsto ((u_1, v_1, \zeta, \xi), t(u_1, v_1, \zeta, \xi)). \end{aligned}$$

Let us see the behaviour of the transition function on the intersection $|U_{00}| \cap |U_{11}|$. A straightforward calculation shows that (see also Chapter 1, Section 1.5):

$$\begin{pmatrix} u_0 & \eta \\ \nu & v_0 \end{pmatrix}^{-1} = \begin{pmatrix} (u_0 - \eta v_0^{-1} \nu)^{-1} & -u_0^{-1} \eta (v_0 - \nu u_0^{-1} \eta)^{-1}) \\ -v_0^{-1} \nu (u_0 - \eta v_0^{-1} \nu)^{-1} & (v_0 - \nu u_0^{-1} \eta)^{-1} \end{pmatrix}.$$

This allows us to view a generic element in $U_{00}(A)$ as an element in $U_{11}(A)$:

$$\begin{pmatrix} 1 & * & 0 & * \\ u_0 & * & \eta & * \\ 0 & * & 1 & * \\ \nu & * & v_0 & * \end{pmatrix} \mapsto \begin{pmatrix} (u_0 - \eta v_0^{-1} \nu)^{-1} & * & -u_0^{-1} \eta (v_0 - \nu u_0^{-1} \eta)^{-1} & * \\ 1 & * & 0 & * \\ -v_0^{-1} \nu (u_0 - \eta v_0^{-1} \nu)^{-1} & * & (v_0 - \nu u_0^{-1} \eta)^{-1} & * \\ 0 & * & 1 & * \end{pmatrix}.$$

We have that sections transform in the following way:

$$\begin{aligned} t(u_1, v_1, \zeta, \xi) \mapsto & x \cdot t((u_0 - \eta v_0^{-1} \nu)^{-1}, -v_0^{-1} \nu (u_0 - \eta v_0^{-1} \nu)^{-1}, \\ & - u_0^{-1} \eta (v_0 - \nu u_0^{-1} \eta)^{-1}, (v_0 - \nu u_0^{-1} \eta)^{-1}) = s(u_0, v_0, \nu, \eta), \end{aligned}$$

where x is an invertible element in $|U_{00}| \cap |U_{11}|$. This means that in order to have a polynomial function the invertible element x must contain a power of the Berezinian. When we consider this transition function, together with all the requirements of the other variables, which we do not compute here, one sees that u_0 and v_0 appear with exponents with *opposite sign*. This means that one copy of $\mathbb{P}^1$ can get embedded (the one corresponding to the positive sign) while the other cannot.

With this particular affine covering, X cannot have any projective embedding since its reduced variety X_0 does not admit such embedding.

In the course of our calculation, we have computed the generic element in $H^1(\mathcal{U}, \mathcal{O}_X^*)$ using the covering $\mathcal{U} = \{U_{00}, U_{01}, U_{10}, U_{11}\}$ of $|X|$ and Čech cohomology. Since $H^1(\mathcal{U}, \mathcal{O}_X^*) = H^1(|X|, \mathcal{O}_X^*)$, the argument we give for the special covering is actually a generic argument and shows that the image of the map ϕ consists only of line bundles with no global sections, hence they will not correspond to line bundles that classically projectively embed into $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Hence $X = \mathrm{Gr}(1|1, 2|2)$ does not admit any projective embedding.

10.6 The infinitesimal theory

In this section we want to discuss the infinitesimal theory of superschemes, that is we define the notion of *tangent space* to a superscheme and to a supervariety at a point of the underlying topological space. We then use these definitions in explicit computations.

Let k be a field. We want to restrict our attention to the algebraic superschemes.

Definition 10.6.1. Let $X = (|X|, \mathcal{O}_X)$ be a superscheme (supervariety). We say that X is *algebraic* if it admits an open affine finite cover $\{X_i\}_{i \in I}$ such that $\mathcal{O}_X(X_i)$ is a finitely generated superalgebra for each X_i.

Unless otherwise specified, in this section all superschemes are assumed to be algebraic.

Given a superscheme $X = (|X|, \mathcal{O}_X)$ each point of x in the topological space $|X|$ belongs to an open affine subsuperscheme $\underline{\mathrm{Spec}}\, A$, $x \cong \mathfrak{p} \in \mathrm{Spec}\, A_0$, so that $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$. Recall that $A_{\mathfrak{p}}$ is the localization of the A_0-module A at the prime ideal $\mathfrak{p} \subset A_0$ and that

$$A_{\mathfrak{p}} = \left\{ \tfrac{f}{g} \mid f \in A,\, g \in A_0 - \mathfrak{p} \right\}.$$

The local ring $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ contains the maximal ideal $m_{X,x}$ generated by the maximal ideal in the local ring $(A_{\mathfrak{p}})_0$ and the generators of $(A_{\mathfrak{p}})_1$ as an A_0-module.

We want to define the notion of a rational point of a superscheme. We will then define the tangent superspace to a superscheme at a rational point.

We want to remark that it is possible to define (as in algebraic geometry) the notion of tangent space at a generic point of a superscheme, not necessarily rational. We however shall restrict our attention to the rational points since, as we shall see, they are in one-to-one correspondence with the k-points of the superscheme and in most applications these are the only interesting points to consider.

Definition 10.6.2. Let $X = (|X|, \mathcal{O}_X)$ be a superscheme. A point $x \in |X|$ is said to be *rational* if $\mathcal{O}_{X,x}/m_{X,x} \cong k$. A point $x \in |X|$ is said to be *closed* if it corresponds to a maximal ideal in Spec A_0, where (Spec $A_0, \mathcal{O}_A) \subset X$ is any affine open neighbourhood of x.

Remark 10.6.3. As in the commutative case we have that if k is algebraically closed, then all closed points of $|X|$ are rational. This is because the field $\mathcal{O}_{X,x}/m_{X,x}$ is a finite algebraic extension of k (see [2], 7.9, and [23], §3, n. 6, for more details).

It is important not to confuse the points of the topological space $|X|$ with the elements in $h_X(A)$ for a generic $A \in$ (salg). These are called *A-points* of the superscheme X. The next observation clarifies the relationship between the points of $|X|$ and h_X, the functor of points of X.

Observation 10.6.4. There is a bijection between the rational points of a superscheme X and the set of its k-points $h_X(k)$. In fact, an element $(|f|, f^*) \in h_X(k)$, $|f|\colon \operatorname{Spec} k \to |X|$, $f^*\colon \mathcal{O}_{X,x} \to k$, determines immediately a point $x = |f|(0)$, which is rational since $\mathcal{O}_{X,x}/m_{X,x} \cong k$. Vice versa a rational point $x \in |X|$ corresponds to a morphism ϕ, where $\phi^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_{X,x} \cong k$, and we obtain $|\phi|\colon \operatorname{Spec} k \to |X|$ simply by assigning to the only prime (0) the point $x \in |X|$.

Definition 10.6.5. Let A be a superalgebra and M an A-module. Let $D\colon A \to M$ be an additive map with the property $D(a) = 0$ for all $a \in k$. We say that D is a *super derivation* if

$$D(fg) = D(f)g + (-1)^{p(D)p(f)} f D(g), \quad f, g \in A,$$

where p as always denotes the parity.

Definition 10.6.6. Let $X = (|X|, \mathcal{O}_X)$ be a superscheme and let x be a rational point in $|X|$. We define the *tangent space of X at x* to be

$$T_x X = \operatorname{Der}(\mathcal{O}_{X,x}, k),$$

where k is viewed as an $\mathcal{O}_{X,x}$-module via the identification $k \cong \mathcal{O}_{X,x}/m_{X,x}$, with $m_{X,x}$ the maximal ideal in $\mathcal{O}_{X,x}$.

The next proposition gives an equivalent definition for the tangent space.

Proposition 10.6.7. *Let X be a superscheme. Then*

$$T_x X = \operatorname{Der}(\mathcal{O}_{X,x}, k) \cong \underline{\operatorname{Hom}}_{(\mathrm{smod})}(m_{X,x}/m_{X,x}^2, k).$$

Note that $m_{X,x}/m^2_{X,x}$ is an $\mathcal{O}_{X,x}$-supermodule which is annihilated by $m_{X,x}$, hence it is a $k = \mathcal{O}_{X,x}/m_{X,x}$-supermodule i.e., a super vector space.

Proof. Let $D \in \mathrm{Der}(\mathcal{O}_{X,x}, k)$. Since D is zero on k and $\mathcal{O}_{X,x} = k \oplus m_{X,x}$, we have that D is determined by its restriction to $m_{X,x}$, $D|_{m_{X,x}}$. Moreover, since $m_{X,x}$ acts as zero on $k \cong \mathcal{O}_{X,x}/m_{X,x}$, one can check that

$$\psi\colon \mathrm{Der}(\mathcal{O}_{X,x}, k) \to \underline{\mathrm{Hom}}_{(\mathrm{smod})}(m_{X,x}/m^2_{X,x}, k), \quad D \mapsto D|_{m_{X,x}},$$

is well defined.

Now we construct the inverse. Let $\alpha\colon m_{X,x} \to k$, $\alpha(m^2_{X,x}) = 0$. Define

$$D_\alpha\colon \mathcal{O}_{X,x} = k \oplus m_{X,x} \to k, \quad D_\alpha(a, f) = \alpha(f).$$

This is a well-defined superderivation.

Moreover one can check that the map $\alpha \mapsto D_\alpha$ is ψ^{-1}. □

The next proposition provides a characterization of the tangent space that is useful for explicit calculations.

Proposition 10.6.8. *Let $X = (|X|, \mathcal{O}_X)$ be a supervariety $x \in |X|$ a rational closed point. Let U be an affine neighbourhood of x, $m_x \subset \mathcal{O}_X(U)$ the maximal ideal corresponding to x. Then*

$$T_x X \cong \underline{\mathrm{Hom}}_{(\mathrm{smod})}(m_{X,x}/m^2_{X,x}, k) \cong \underline{\mathrm{Hom}}_{(\mathrm{smod})}(m_x/m^2_x, k).$$

Proof. The proof is the same as in the ordinary case and is based on the fact that localization commutes with exact sequences. □

Let us compute explicitly the tangent space in an example.

Example 10.6.9. Consider the affine supervariety represented by the coordinate ring

$$\mathbb{C}[x, y, \xi, \eta]/(x\xi + y\eta).$$

Notice that the reduced variety is the affine plane.

Since $\mathbb{C}$ is algebraically closed, all closed points are rational. Consider the closed point $P = (1, 1, 0, 0) \cong m_P = (x-1, y-1, \xi, \eta) \subset \mathbb{C}[x, y, \xi, \eta]/(x\xi + y\eta)$, where we identify $(x_0, y_0, 0, 0)$ with maximal ideals in the ring of the supervariety, as we do in the commutative case. By Proposition 10.6.8, the tangent space at P consists of all the functions $\alpha\colon m_P \to k$, $\alpha(m^2_P) = 0$.

A generic $f \in m_P$ lifts to the family of $f = f_1 + f_2(x\xi + y\eta) \in \mathbb{C}[\mathbb{A}^{2|2}] = \mathbb{C}[x, y, \xi, \eta]$ with $f_1(1, 1, 0, 0) = 0$ and where f_2 is any function in $\mathbb{C}[\mathbb{A}^{2|2}] = \mathbb{C}[x, y, \xi, \eta]$. Thus f can be formally expanded in power series around P:

$$f = \tfrac{\partial f_1}{\partial x}(P)(x-1) + \tfrac{\partial f_1}{\partial y}(P)(y-1) + (\tfrac{\partial f_1}{\partial \xi}(P) + f_2(P))\xi$$
$$+ (\tfrac{\partial f_1}{\partial \eta}(P) + f_2(P))\eta + \text{higher order terms}.$$

Define

$$X = \frac{\partial f_1}{\partial x}(P), \quad Y = \frac{\partial f_1}{\partial y}(P), \quad \Xi = \frac{\partial f_1}{\partial \xi}(P), \quad \mathrm{E} = \frac{\partial f_1}{\partial \eta}(P).$$

These are coordinates for the super vector space $\mathbb{M}_P/\mathbb{M}_P^2$, $\mathbb{M}_P = (x-1, y-1, \xi, \eta) \subset \mathbb{C}[x, y, \xi, \eta]$. A basis for the dual space $(\mathbb{M}_p/\mathbb{M}_p^2)^*$, which is the tangent space to $\mathbb{A}^{2|2}$, consists of functions sending the coefficient of one of the $x-1$, $y-1$, ξ, η to a non-zero element and the others to zero. We get equations for the tangent space $(m_P/m_P^2)^*$ as a subspace of $(\mathbb{M}_P/\mathbb{M}_P^2)^*$:

$$\Xi - \mathrm{E} = 0.$$

So we have described the tangent space $(m_P/m_P^2)^*$ as a subspace of $(\mathbb{M}_P/\mathbb{M}_P^2)^*$, the tangent space to the affine superspace $\mathbb{A}^{m|n}$.

There is yet another way to compute the tangent space, in the case X is an affine supervariety. Before we examine this construction, we must understand first the notion of *differential of a function* and *differential of a morphism*. We start by defining the value and the differential for the germs of sections.

Definition 10.6.10. Let $X = (|X|, \mathcal{O}_X)$ be a superscheme, x a rational point.

Consider the projections

$$\pi\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_{X,x} \cong k, \quad p\colon m_{X,x} \to m_{X,x}/m_{X,x}^2.$$

Let $f \in \mathcal{O}_{X,x}$. We define the *value of* f *at* x to be

$$f(x) := \pi(f).$$

Notice that $f - f(x) \in m_{X,x}$, where we interpret $f(x) \in k \subset \mathcal{O}_{X,x}$. We also define the *differential of* f *at* x to be

$$(df)_x := p(f - f(x)).$$

We now want to define the value and differential of a section at a point.

If U is an open neighbourhood of x and $f \in \mathcal{O}_X(U)$, we define the *value of* f *at* x to be

$$f(x) := \pi(\phi(f)),$$

where $\phi\colon \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is the natural morphism.

Finally, the *differential of* f *at* x is

$$(df)_x := (d\phi(f))_x.$$

We leave to the reader the simple check that this definition is independent from the chosen open neighbourhood U of x.

Example 10.6.11. If $P = (x_1^0, \dots, x_m^0, 0, \dots, 0)$ is a closed rational point of the affine superspace $\mathbb{A}^{m|n}$ with coordinate ring $k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$, a basis of m_P/m_P^2 is $\{x - x_i^0, \xi_j\}_{\{i=1,\dots,m, j=1,\dots,n\}}$. Hence one can readily see that

$$(dx_i)_P = x - x_i^0, \quad (d\xi_j)_P = \xi_j, \quad i = 1, \dots, m, \ j = 1, \dots, n.$$

Definition 10.6.12. Let $(|\alpha|, \alpha^*)\colon X \to Y$ be a morphism of superschemes and x a rational point in $|X|$. Then $|\alpha|(x)$ is also rational and α induces a morphism $d\alpha_x\colon T_xX \to T_{|\alpha|(x)}Y$ by

$$d\alpha_x(D)f = D(\alpha_x^*(f)), \quad D \in T_xX = \mathrm{Der}(\mathcal{O}_{X,x}, k), \quad \alpha_x^*\colon \mathcal{O}_{Y,|\alpha|(x)} \to \mathcal{O}_{X,x},$$

called the *differential of* α at the point $x \in |X|$.

Definition 10.6.13. We say that $\alpha\colon X \to Y$ is a *closed embedding* if $|\alpha|$ is a homeomorphism of $|X|$ onto a closed subset of $|Y|$ and α^* is a surjective sheaf morphism.

If X and Y are affine superschemes, $X = \underline{\mathrm{Spec}}\, A$, $Y = \underline{\mathrm{Spec}}\, B$, then α is a closed embedding if and only if α^* is surjective on the global sections, that is, $\alpha^*\colon A \to B \cong A/I$ for some ideal I in A.

Proposition 10.6.14. *If $(|\alpha|, \alpha^*)$ is a closed embedding, then $d\alpha_x$ is injective.*

Proof. Direct check. □

If X is a subsupervariety of $\mathbb{A}^{m|n}$ it makes sense to ask for equations that determine the tangent superspace to X as a linear subsuperspace of $T_x\mathbb{A}^{m|n} \cong k^{m|n}$, as we did in Example 10.6.9 for a special case.

Proposition 10.6.15. *Let X be a subvariety of $\mathbb{A}^{m|n}$ defined by the ideal $I \subset \mathcal{O}(\mathbb{A}^{m|n}) = k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$. Let x be a rational closed point of X. Then*

$$T_xX = \{v \in k^{m|n} \mid (df)_x(v) = 0 \textit{ for all } f \in I\}.$$

Proof. The closed embedding $\alpha\colon X \subset \mathbb{A}^{m|n}$ corresponds to a surjective morphism $\phi\colon \mathcal{O}(\mathbb{A}^{m|n}) \to \mathcal{O}(X)$, hence $\mathcal{O}(X) \cong \mathcal{O}(\mathbb{A}^{m|n})/I$. Let m_x and $\mathbb{M}_x$ denote respectively the maximal ideal associated to x in X and $\mathrm{Spec}\, \mathcal{O}(\mathbb{A}^{m|n})_0$, respectively. The map ϕ induces a surjective linear map

$$\psi\colon \mathbb{M}_x/\mathbb{M}_x^2 \twoheadrightarrow m_x/m_x^2.$$

between superspaces. Let us recall the following simple fact of linear algebra.

If $a\colon V_1 \to V_2$ is a surjective linear map between finite-dimensional vector spaces V_1, V_2 and $b\colon V_2^* \subset V_1^*$ is the injective linear map induced by a on the dual vector spaces, then $s \in \mathrm{im}(b)$ if and only if $s|_{\ker(a)} = 0$.

We apply this observation to the maps ψ and the differential $(d\alpha)_x$,

$$(d\alpha)_x\colon T_x(X) = (m_x/m_x^2)^* \hookrightarrow T_{\alpha(x)}(\mathbb{A}^{m|n}) = (\mathbb{M}_x/\mathbb{M}_x^2)^*,$$

and we see that

$$T_x(X) = \{v \in T_{|\alpha|(x)}(\mathbb{A}^{m|n}) \mid v(\ker(\psi)) = 0\}.$$

Observe that $\ker(\psi) = \{(df)_x \mid f \in I\}$. By identifying $\mathbb{A}^{m|n} = k^{m|n}$ with its double dual $(k^{m|n})^{**}$ we obtain the result. □

Remark 10.6.16. In the notation of the previous proposition, if $I = (f_1, \dots, f_r)$ one can readily check that

$$T_x X = \{v \in k^{m|n} \mid (df_i)_x(v) = 0 \text{ for all } i = 1, \dots, r\},$$

thus obtaining a quick and effective method to calculate the tangent space to a supervariety.

Let us revisit Example 10.6.9 and see how the calculation is made using Proposition 10.6.15.

Example 10.6.17. Consider again the supervariety represented by the superalgebra

$$\mathbb{C}[x, y, \xi, \eta]/(x\xi + y\eta).$$

We want to compute the tangent space at $P = (1, 1, 0, 0) = (x_0, y_0, \xi_0, \eta_0)$:

$$\begin{aligned} d(x\xi + y\eta)_P &= x_0(d\xi)_P + \xi_0(dx)_P + y_0(d\eta)_P + \eta_0(dy)_P \\ &= (d\xi)_P + (d\eta)_P \\ &\cong (0, 0, 1, 1). \end{aligned}$$

Hence by Proposition 10.6.15 the tangent space is the subspace of $k^{2|2}$ given by the equation

$$\xi - \eta = 0.$$

10.7 References

The definition of superscheme together with its functor of points appears in the work of Manin [56], where also the examples of the Grassmannian and flag superschemes are described in detail in the complex analytic category. The representability criterion Theorem 10.3.7 appears in [23], Ch. I, for the ordinary setting and is used, however not formally proved, in the work by Manin [56].

11

Algebraic supergroups

In this chapter we restrict our attention to the superschemes that have an additional structure, namely the group structure, and thus are called *supergroup schemes* or simply supergroups for short. The simplest way to introduce this extra structure is the requirement for the functor of points to be group-valued, that is, if X is a superscheme, we are asking that $h_X(A)$ be a group for each superalgebra A.

As a general rule in supergeometry, the functor of points is valued in the ordinary categories like sets, groups, vector spaces or Lie algebras and the supernature of the geometrical object stems from the category we start from, in our case superalgebras or superschemes. As it happens for the ordinary setting, when the superscheme G is affine, G is a supergroup if and only if its representing superalgebra $\mathcal{O}(G)$ is a *Hopf superalgebra*. We shall discuss in detail the example of the general and special linear supergroups together with their Hopf superalgebras.

As for the ordinary setting, we can associate very naturally to any supergroup its *Lie superalgebra*, which is a Lie algebra-valued functor, identified with the functor of points of the super tangent space of the supergroup at the identity. We also introduce the concept of representation of a supergroup and we prove the following two important results: we show that any affine supergroup can be embedded into some $\mathrm{GL}(V)$ for a suitable V, and then we prove the representability of the stabilizer functor for the action of a supergroup on a superscheme. This gives the representability of the classical algebraic supergroups corresponding to the classical Lie superalgebras (see Appendix A for their list).

Our treatment follows very closely [23], Ch. II; most of the classical statements go unchanged to the super-setting and we shall point out the differences when they arise.

11.1 Supergroup functors and supergroup schemes

Let k be a commutative ring. All superalgebras are assumed to be commutative and over k unless otherwise specified. Their category is denoted by (salg).

A *supergroup scheme* is a superscheme whose functor of points is group-valued, that is to say, it associates a group to each superscheme or equivalently to each superalgebra. In fact, as we know, the functor of points of a superscheme is determined by its restriction to affine superschemes, whose category is equivalent to the category of superalgebras.

As often happens in algebraic geometry, in order to study supergroup schemes we need first to define and understand the weaker notion of *supergroup functor*, which is simply a group-valued functor, from (salg) to (sets), without any representability requirement that characterizes the functor of points of supergroup schemes and more in general superschemes.

Definition 11.1.1. A *supergroup functor* is a group-valued functor

$$G\colon (\mathrm{salg}) \to (\mathrm{sets}),$$

where by group-valued functor we mean a functor valued in the category of groups.

This is equivalent to having the following natural transformations:

(1) Multiplication $\mu\colon G \times G \to G$ such that $\mu \circ (\mu \times \mathrm{id}) = (\mu \times \mathrm{id}) \circ \mu$, i.e.,

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \mathrm{id}} & G \times G \\ {\scriptstyle \mathrm{id}\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G. \end{array}$$

(2) Unit $e\colon e_k \to G$, where $e_k\colon (\mathrm{salg}) \to (\mathrm{sets})$, $e_k(A) = 1_A$ (in other words, $e_k = h_{\underline{\mathrm{Spec}}\, k}$) such that $\mu \circ (\mathrm{id} \times e) = \mu \circ (e \times \mathrm{id})$, i.e.,

$$\begin{array}{ccccc} G \times e_k & \xrightarrow{\mathrm{id}\times e} & G \times G & \xleftarrow{e \times \mathrm{id}} & e_k \times G \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & G. & & \end{array}$$

(3) Inverse $i\colon G \to G$ such that $\mu \circ (\mathrm{id}, i) = e \circ \mathrm{id}$, i.e.,

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id}, i)} & G \times G \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ e_k & \xrightarrow{e} & G. \end{array}$$

The supergroup functors together with their morphisms, that is the natural transformations that preserve μ, e and i, form a category.

If G is the functor of points of a superscheme X, i.e., $G = h_X$, in other words $G(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, X)$, we say that X is a *supergroup scheme*. An *affine supergroup scheme* X is a supergroup scheme which is an affine superscheme, that is $X = \underline{\mathrm{Spec}}\, \mathcal{O}(X)$ for some superalgebra $\mathcal{O}(X)$. To make the terminology easier we will drop the word "scheme" when speaking of supergroup schemes, whenever there is no danger of confusion.

As we shall presently see, the functor of points of an affine supergroup is represented by a superalgebra, which has the additional structure of a *Hopf superalgebra*. We refer the reader to Chapter 1, Section 1.7, for their definitions and main properties.

Proposition 11.1.2. *Let G be an affine superscheme. Then G is a supergroup if and only if $\mathcal{O}(G)$ is a super Hopf algebra. Moreover, we can identify the category of affine supergroups with the category of commutative Hopf superalgebras.*

Proof. We first observe that, if G is a superscheme, and $\mathcal{O}(G)$ is a Hopf superalgebra with comultiplication Δ, counit ϵ and antipode S, $h_G(A)$ has a natural group structure. In fact we can define the product of two morphisms in $h_G(A)$ by

$$x \cdot y = \mu_A \circ x \otimes y \circ \Delta\colon \mathcal{O}(G) \xrightarrow{\Delta} \mathcal{O}(G) \otimes \mathcal{O}(G) \xrightarrow{x \otimes y} A \otimes A \xrightarrow{\mu_A} A,$$

where μ_A is the multiplication in the superalgebra A. One can immediately check that the multiplication is a morphism, that is,

$$(x \cdot y)(ab) = (x \cdot y)(a)(x \cdot y)(b) \quad \text{for all } a, b \in A$$

(though hidden, the sign rule plays a crucial role here). This multiplication in $h_G(A)$ gives rise to Δ in $\mathcal{O}(G)$ as its associated comultiplication, as one can readily see.

The unit e_A and the inverse i_A in $h_G(A)$ are defined by

$$e_A = \eta_A \circ \epsilon\colon \mathcal{O}(G) \xrightarrow{\epsilon} k \xrightarrow{\eta_A} A, \quad i_A(x) = x \circ S,$$

where η_A is the unit in A. We leave to the reader the routine checks of Definition 11.1.1.

Vice versa, if G is a supergroup scheme, then we can define the comultiplication $\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ as the dual of the multiplication $\mu \in \mathrm{Hom}(G \times G, G)$ using the identification

$$\mathrm{Hom}(G \times G, G) \cong \mathrm{Hom}(\mathcal{O}(G), \mathcal{O}(G \times G))$$

(one can readily check that $\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$). Similarly one defines the counit and the antipode ϵ and S as the duals of unit e and inverse i.

A careful look shows that formally the diagrams defining a supergroup functor are essentially the same as those defining a Hopf superalgebra, with arrows reversed. We leave to the reader the routine verification that h_G satisfies all the diagrams in Definition 11.1.1 if and only if $\mathcal{O}(G)$ satisfies the diagrams in Definition 1.7.1.

The equivalence between the categories of affine supergroups and commutative Hopf superalgebras is an immediate consequence of the previous discussion and Proposition 10.1.9. □

Let us now examine some important examples of supergroup schemes and their associated Hopf superalgebras.

Examples 11.1.3. (1) *Supermatrices* $\mathrm{M}_{m|n}$. In Remark 1.4.1 we have introduced the functor of points of supermatrices

$$\mathrm{M}_{m|n}\colon (\text{salg}) \to (\text{sets}), \quad A \mapsto \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix},$$

where a and b are $m \times m$, $n \times n$ block matrices with entries in A_0, while α and β are $m \times n$, $n \times m$ block matrices with entries in A_1. As we have seen in Example 3.3.4, $\mathrm{M}_{m|n}$ is a representable functor, represented by the superalgebra of polynomials $k[x_{ij}, \xi_{kl}]$ for suitable indices i, j, k, l. The functor $\mathrm{M}_{m|n}$ is group-valued, in fact any $\mathrm{M}_{m|n}(A)$ has an additive group structure, where the addition is simply defined as the addition of matrices. Hence by the previous proposition $k[x_{ij}, \xi_{kl}]$ is a Hopf superalgebra, where the comultiplication Δ, the counit ϵ and antipode S are given by

$$\Delta(x_{ij}) = x_{ij} \otimes 1 + 1 \otimes x_{ij}, \quad \Delta(\xi_{kl}) = \xi_{kl} \otimes 1 + 1 \otimes \xi_{kl},$$
$$\epsilon(x_{ij}) = 0, \quad \epsilon(\xi_{ij}) = 0, \quad S(x_{ij}) = -x_{ij}, \quad S(\xi_{ij}) = -\xi_{ij}.$$

We leave to the reader the verification that $k[x_{ij}, \xi_{kl}]$ together with Δ, ϵ and S is a Hopf superalgebra.

(2) *The general linear supergroup* $\mathrm{GL}_{m|n}$. Let $A \in (\text{salg})$. Let us define $\mathrm{GL}_{m|n}(A)$ as $\mathrm{GL}(A^{m|n})$ (see Chapter 1) to be the set of automorphisms of the A-supermodule $A^{m|n}$. Choosing the standard basis we can write

$$\mathrm{GL}_{m|n}(A) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \right\} \subset \mathrm{M}_{m|n}(A).$$

As we have previously remarked, $\mathrm{GL}_{m|n}(A)$ are the invertible transformations of $k^{m|n}(A)$ preserving parity.

This is the functor of points of an affine supergroup $\mathrm{GL}_{m|n}$ represented by the Hopf superalgebra

$$\mathcal{O}(\mathrm{GL}_{m|n}) = k[x_{ij}, \xi_{kl}][U, V]/(Ud_1 - 1, Vd_2 - 1)$$

where x_{ij}'s, U, V and ξ_{kl}'s are respectively even and odd variables with $1 \le i, j \le m$ or $m+1 \le i, j \le m+n$, $1 \le k \le m$, $m+1 \le l \le m+n$ or $m+1 \le k \le m+n$, $1 \le l \le m$ and

$$d_1 = \sum_{s \in S_m} (-1)^{l(s)} x_{1,s(1)} \dots x_{m,s(m)},$$
$$d_2 = \sum_{t \in S_n} (-1)^{l(t)} x_{m+1,m+t(1)} \dots x_{m+n,m+t(n)}.$$

It is customary to write d_1^{-1} and d_2^{-1} in place of U and V, so we shall note

$$\mathcal{O}(\mathrm{GL}_{m|n}) = k[x_{ij}, \xi_{kl}][d_1^{-1}, d_2^{-1}].$$

Notice that the Berezinian function is well defined in $\mathcal{O}(\mathrm{GL}_{m|n})$, in fact

$$\mathrm{Ber} = d_2^{-1} \det(a - \beta b^{-1} \alpha).$$

Since $\mathrm{GL}_{m|n}$ is a representable group-valued functor, Proposition 11.1.2 ensures that there is a well-defined Hopf superalgebra structure on $\mathcal{O}(\mathrm{GL}_{m|n})$, corresponding (dually) to the multiplicative group structure on each $\mathrm{GL}_{m|n}(A)$. Nevertheless, we want to explicitly describe such a Hopf superalgebra structure.

Let us first start with the bialgebra structure of $\mathcal{O}(\mathrm{GL}_{m|n})$.

The comultiplication Δ and the counit ϵ are given by the following formulas. To ease the notation, let $a_{ij} = x_{ij}$ if $p(i) + p(j)$ is even and $a_{ij} = \xi_{ij}$ otherwise (an index is even if it is between 1 and m). We define

$$\Delta(a_{ij}) = \sum_{k=1}^{m+n} a_{ik} \otimes a_{kj}, \quad \epsilon(a_{ij}) = \delta_{ij}.$$

We need to specify Δ, ϵ and S also on the generators d_1^{-1} and d_2^{-1} and then to check that they are well defined with respect to the ideal of the relations $d_1^{-1}d_1 = 1$, $d_2^{-1}d_2 = 1$:

$$\Delta(d_1^{-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1}(d_1^{-1})^i \otimes (d_1^{-1})^i(\Delta(d_1) - d_1^{-1} \otimes d_1^{-1})^{i-1},$$
$$\Delta(d_2^{-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1}(d_2^{-1})^i \otimes (d_2^{-1})^i(\Delta(d_2) - d_2^{-1} \otimes d_2^{-1})^{i-1},$$
$$\epsilon(d_1^{-1}) = \epsilon(d_2^{-1}) = 1.$$

We leave to the reader the tedious verification that Δ, ϵ are well defined and satisfy respectively the properties of the comultiplication and counit as in Definition 1.7.1 and they correspond to the group structure of the functor $\mathrm{GL}_{m|n}$.

As for the antipode, the definition turns out to be more complicated and this is a consequence of the fact that the inverse for a supermatrix has a far more complicated formula than the inverse of an ordinary matrix. For such formulas, we refer the interested reader to [30].

(3) *The special linear supergroup* $\mathrm{SL}_{m|n}$. For a superalgebra A, let us define $\mathrm{SL}_{m|n}(A)$ to be the subgroup of $\mathrm{GL}_{m|n}(A)$ consisting of matrices with Berezinian equal to 1. This is the functor of points of an affine supergroup and it is represented by the Hopf superalgebra

$$\mathcal{O}(\mathrm{SL}_{m|n}) = k[x_{ij}, \xi_{kl}][d_1^{-1}, d_2^{-1}]/(\mathrm{Ber} - 1),$$

where the comultiplication, counit and antipode are inherited naturally from the ones in $\mathrm{GL}_{m|n}$.

We end our introduction on supergroup schemes with an observation concerning the unit of a supergroup scheme that we shall need later.

Observation 11.1.4. When k is a field, we may interpret the unit e of a supergroup scheme $G = (|G|, \mathcal{O}_G)$ as a rational point, denoted by 1_G, of G. This is a point in $|G|$

for which $\mathcal{O}_{G,1_G}/m_{1_G} \cong k$, where m_{1_G} is the maximal ideal in the local superring $\mathcal{O}_{G,1_G}$, in the following way. By definition, the unit e is a morphism of superschemes $e\colon e_k \to G$, hence corresponds to a pair of morphisms $e = (|e|, e^*)$, $|e|\colon |e_k| \to |G|$, $e^*\colon \mathcal{O}_G \to e_*\mathcal{O}_{e_k} = k$. Define $1_G := |e|(|e_k|) \in |G|$. This is a rational point, in fact $\mathcal{O}_{G,1_G}/m_{1_G} \cong k$. Moreover notice that, by the very definition of e, 1_G has the property of a unit for the topological group $|G|$.

11.2 Lie superalgebras

In Chapter 1 we defined a Lie superalgebra as a super vector space with a bracket satisfying the antisymmetry property and the Jacobi identity, of course with suitable signs. In this section we want to reformulate functorially the same notion.

If $\mathfrak{g}$ is a finite-dimensional super vector space and $h_\mathfrak{g}$ is its functor of points, we shall see that $\mathfrak{g}$ is a Lie superalgebra if and only if $h_\mathfrak{g}$ is Lie algebra-valued, in other words, $h_\mathfrak{g}(A)$ is a Lie superalgebra for all superalgebras A. As we remarked at the beginning of this chapter, the reader should not be confused by the fact that $h_\mathfrak{g}$ is valued in the category of ordinary Lie algebras. This is in complete analogy with the fact that the functor of points of a supergroup is a group-valued functor. The supergeometric nature of these objects is expressed by the category we start from, namely the category of superalgebras.

Consider the functor of points of the superscheme $\mathbb{A}^{1|0}$, the affine line, $\mathcal{O}_k\colon$ (salg) $\to$ (sets), $\mathcal{O}_k(A) = \mathrm{Hom}(k[x], A) \cong A_0$ For notational purposes we use the symbol $\mathcal{O}_k$ to denote it, instead of $h_{\mathbb{A}^{1|0}}$.

Definition 11.2.1. Let M be a functor $M\colon$ (salg) $\to$ (sets), with an operation, that is for each $A \in$ (salg) we define functorially the operation $M(A) \times M(A) \to M(A)$, $(x, y) \mapsto xy$, sometimes also denoted additively as $(x, y) \mapsto x + y$. We say that M is an $\mathcal{O}_k$*-module* if we have a natural transformation

$$\mathcal{O}_k(A) \times M(A) \to M(A), \quad (a, x) \mapsto ax,$$

such that

- $a(xy) = (ax)(ay)$,
- $(ab)x = a(bx)$

for all $a, b \in \mathcal{O}_k(A)$ and $x, y \in M(A)$,

For example if M is the functor of points of a super vector space V, that is, $M(A) = (A \otimes V)_0$, we can define the operation $(x, y) \mapsto x + y$ that associates to a pair of elements in the A_0-module $M(A)$ their sum. Then M has a natural structure of an $\mathcal{O}_k$-module, which is the multiplication of elements in $M(A)$ by scalars in A_0.

Definition 11.2.2. Let $\mathfrak{g}$ be a super vector space. We say that the functor

$$L_{\mathfrak{g}}: (\text{salg}) \to (\text{sets}), \quad L_{\mathfrak{g}}(A) = (A \otimes \mathfrak{g})_0,$$

is *Lie algebra-valued* if it is an $\mathcal{O}_k$-module and there is an $\mathcal{O}_k$-linear natural transformation, called the *bracket*,

$$[\ ,\ ]: L_{\mathfrak{g}} \times L_{\mathfrak{g}} \to L_{\mathfrak{g}}$$

that satisfies commutative diagrams corresponding to the ordinary antisymmetric property and the Jacobi identity. In other words, for each superalgebra A, there is a well-defined bracket $[\ ,\ ]_A$ that defines functorially a Lie algebra structure on the A_0-module $L_{\mathfrak{g}}(A) = \mathfrak{g}(A)$. We will drop the suffix A from the bracket and the natural transformations to ease the notation.

Remark 11.2.3. If the super vector space $\mathfrak{g}$ is finite-dimensional, then the functor $L_{\mathfrak{g}}$ is representable and we have $L_{\mathfrak{g}} = h_{\mathfrak{g}}$. In fact

$$L_{\mathfrak{g}}(A) = (A \otimes \mathfrak{g})_0 = \text{Hom}_{(\text{smod})}(\mathfrak{g}^*, A) = \text{Hom}_{(\text{salg})}(\text{Sym}(\mathfrak{g}^*), A),$$

where (smod) denotes the category of supermodules (over k in this case) and $\text{Sym}(\mathfrak{g}^*)$ the symmetric algebra over $\mathfrak{g}^*$. Notice also that in this case $L_{\mathfrak{g}}$ is the functor of points of an affine superscheme represented by the superalgebra $\text{Sym}(\mathfrak{g}^*)$.

We now want to see that the usual notion of Lie superalgebra, as we defined in Chapter 1, Section 1.2, is equivalent to this functorial definition. We want to show that $\mathfrak{g}$ is a Lie superalgebra if and only if the functor $L_{\mathfrak{g}}$ is Lie algebra-valued.

Let us first recall the definition of Lie superalgebra given in Chapter 1, Section 1.2.

Definition 11.2.4. Let $\mathfrak{g}$ be a super vector space. We say that $\mathfrak{g}$ is a Lie superalgebra if there exists a bilinear map $[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called a *superbracket* such that

(a) $[x, y] = (-1)^{p(x)p(y)}[y, x]$,

(b) $[x,[y,z]] + (-1)^{p(x)p(y)+p(x)p(z)}[y,[z,x]] + (-1)^{p(x)p(z)+p(y)p(z)}[z,[x,y]]$

for all $x, y, z \in \mathfrak{g}$.

Proposition 11.2.5. *Let $\mathfrak{g}$ be a super vector space. Then $\mathfrak{g}$ is a Lie superalgebra if and only if $L_{\mathfrak{g}}: (\text{salg}) \to (\text{sets})$, $L_{\mathfrak{g}}(A) = (A \otimes \mathfrak{g})_0$, is a Lie algebra-valued functor.*

Proof. This is an immediate consequence of the even rules principle, detailed in Chapter 1, Section 1.8. Nevertheless, given the importance of this construction, we want to see explicitly how the superbracket on $\mathfrak{g}$ and the bracket on $L_{\mathfrak{g}}(A)$ correspond to each other.

If we have a Lie superalgebra $L_{\mathfrak{g}}$ there is always, by definition, a super vector space $\mathfrak{g}$ associated to it. Moreover by the even rules, there is a unique Lie superalgebra structure on the A-module $L_{\mathfrak{g}}^A = A \otimes \mathfrak{g}$, whose even part is the Lie algebra $L_{\mathfrak{g}}(A)$.

Given $v, w \in \mathfrak{g}$, since the Lie bracket on $L_{\mathfrak{g}}^A$ is A-linear, we can define the element $\{v, w\} \in \mathfrak{g}$ as

$$[a \otimes v, b \otimes w] = (-1)^{p(b)p(v)} ab \otimes \{v, w\} \in (A \otimes \mathfrak{g})_0 \in L_{\mathfrak{g}}(A),\ a \otimes v, b \otimes w \in (A \otimes \mathfrak{g})_0.$$

Clearly the bracket $\{v, w\} \in \mathfrak{g}$ does not depend on $a, b \in A$. One can verify that it is a superbracket. Let us see, for example, the antisymmetry property. Observe first that if $a \otimes v \in (A \otimes \mathfrak{g})_0$, then $p(v) = p(a)$ since $(A \otimes \mathfrak{g})_0 = A_0 \otimes \mathfrak{g}_0 \oplus A_1 \otimes \mathfrak{g}_1$. So we can write

$$[a \otimes v, b \otimes w] = (-1)^{p(b)p(v)} ab \otimes \{v, w\}.$$

On the other hand,

$$\begin{aligned}[b \otimes w, a \otimes v] &= (-1)^{p(a)p(w)} ba \otimes \{w, v\} \\ &= (-1)^{p(a)p(w)+p(a)p(b)} ab \otimes \{w, v\} \\ &= (-1)^{2p(w)p(v)} ab \otimes \{w, v\} \\ &= ab \otimes \{w, v\}.\end{aligned}$$

By comparing the two expressions we get the antisymmetry of the superbracket. For the super Jacobi identity the calculation is the same.

A similar calculation also shows that, given a super Lie algebra $\mathfrak{g}$, the functor $L_{\mathfrak{g}}$ is Lie algebra-valued. □

Hence the previous proposition shows that a Lie algebra-valued functor $L_{\mathfrak{g}}$ according to Definition 11.2.2 is equivalent to a super Lie algebra $\mathfrak{g}$. With an abuse of language we will refer to both $\mathfrak{g}$ and $L_{\mathfrak{g}}$ as "Lie superalgebra".

Remark 11.2.6. Given a super vector space $\mathfrak{g}$ one may also define a Lie superalgebra to be the representable functor $D_{\mathfrak{g}} \colon$ (salg) $\to$ (sets) so that

$$D_{\mathfrak{g}}(A) = \mathrm{Hom}_{(\mathrm{smod})}(\mathfrak{g}^*, A) = \mathrm{Hom}_{(\mathrm{salg})}(\mathrm{Sym}(\mathfrak{g}^*), A)$$

with an $\mathcal{O}_k$ linear natural transformation $[\ ,\ ] \colon D_{\mathfrak{g}} \times D_{\mathfrak{g}} \to D_{\mathfrak{g}}$ satisfying the commutative diagrams corresponding to antisymmetry and Jacobi identity. When $\mathfrak{g}$ is finite-dimensional this definition coincides with Definition 11.2.2, however we have preferred the one given there since its immediate equivalence with the definition is the one mostly used in the literature.

Example 11.2.7 (The supermatrices as a Lie algebra-valued functor). Consider again the functor of points of supermatrices

$$A \mapsto \mathrm{M}_{m|n}(A) = \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right\},$$

where P, Q, R, S are respectively $(m \times m)$, $(m \times n)$, $(n \times m)$, $(n \times n)$-matrices with entries: P and S in A_0, R and Q in A_1.

This is a Lie algebra-valued functor once we define a Lie bracket on each $\mathrm{M}_{m|n}(A)$:

$$[X, Y] = XY - YX \quad \text{for all } X, Y \in \mathrm{M}_{m|n}(A).$$

In Example 3.3.4 we have seen that $\mathrm{M}_{m|n}$ is a representable functor corresponding to a super vector space $\mathrm{M}(m|n)$ (by an abuse of notation we may at times use the same letter). The super vector space $\mathrm{M}(m|n)$ is a super Lie algebra with bracket

$$[X, Y] = XY - (-1)^{p(x)p(y)} YX \quad \text{for all } X, Y \in \mathrm{M}(m|n).$$

One word of warning: the super vector space $\mathrm{M}(m|n)$ is *not* $\mathrm{M}_{m|n}(k)$. In fact $\mathrm{M}_{m|n}(k)$ consists only of the even part of the super Lie algebra $\mathrm{M}(m|n)$ and contains just the diagonal block matrices with entries in k, while $\mathrm{M}(m|n)$, as a vector space, consists of $(m + n \times m + n)$-matrices with entries in k and has superdimension $m^2 + n^2|2mn$.

The purpose of the next two sections is to naturally associate a Lie superalgebra $\mathrm{Lie}(G)$ to a supergroup G.

11.3 Lie(G) of a supergroup functor G

In ordinary geometry we can associate to any group scheme G a Lie algebra, commonly denoted by $\mathrm{Lie}(G)$, which is identified with the tangent space to the group scheme G at the identity. This is an extremely important construction since it allows us to linearize problems, by transferring our questions from the group to its Lie algebra.

We want to proceed and repeat the same construction in the supergeometric setting and we shall achieve this in two steps: first we define the super vector space $\mathrm{Lie}(G)$ associated to the supergroup G and we show that it is isomorphic to the tangent space to G at the identity element. Then, in the next section, we show that $\mathrm{Lie}(G)$ is a Lie superalgebra functor. Our treatment follows closely the classical one as detailed in [23], Ch. II.

As customary in algebraic geometry and in supergeometry, we start by giving the more general notion of $\mathrm{Lie}(G)$ associated to a supergroup functor G (not necessarily representable).

Let G be a supergroup functor. Let A be a commutative superalgebra and let $A(\epsilon) := A[\epsilon]/(\epsilon^2)$ be the algebra of dual numbers, where ϵ is an *even* indeterminate. We have $A(\epsilon) = A \oplus \epsilon A$, and there are two natural morphisms

$$i\colon A \to A(\epsilon),\ i(1) = 1, \quad p\colon A(\epsilon) \to A,\ p(1) = 1,\ p(\epsilon) = 0,\ p \circ i = \mathrm{id}_A.$$

Definition 11.3.1. Consider the homomorphism $G(p)\colon G(A(\epsilon)) \to G(A)$. For each G there is a supergroup functor

$$\mathrm{Lie}(G)\colon (\mathrm{salg}) \to (\mathrm{sets}), \quad \mathrm{Lie}(G)(A) := \ker(G(p)).$$

If G is a supergroup scheme, we denote $\mathrm{Lie}(h_G)$ by $\mathrm{Lie}(G)$.

If $f : G \to H$ is a natural transformation of supergroup functors, we have the following commutative diagram (where the vertical arrows form exact sequences):

$$
\begin{array}{ccc}
1 & \longrightarrow & 1 \\
\uparrow & & \uparrow \\
G(A) & \xrightarrow{f_A} & H(A) \\
{\scriptstyle G(p)}\uparrow & & \uparrow{\scriptstyle H(p)} \\
G(A(\epsilon)) & \xrightarrow{f_{A(\epsilon)}} & H(A(\epsilon)) \\
\uparrow & & \uparrow \\
\mathrm{Lie}(G)(A) & \xrightarrow{f_{A(\epsilon)}|_{\mathrm{Lie}(G)(A)}} & \mathrm{Lie}(H)(A) \\
\uparrow & & \uparrow \\
1 & \longrightarrow & 1.
\end{array}
$$

We hence define: $\mathrm{Lie}(f)(A) = f_{A(\epsilon)}|_{\mathrm{Lie}(G)(A)}$. The following proposition is immediate.

Proposition 11.3.2. Lie *is a functor from the category of supergroup functors to the category of groups.*

We are going to show that, when G is an algebraic supergroup scheme, $\mathrm{Lie}(G)$ is a Lie algebra-valued functor, thus associating to any algebraic supergroup scheme a Lie superalgebra. Let us first examine some important examples.

Examples 11.3.3. (1) *The general linear Lie superalgebra.* We want to determine the functor $\mathrm{Lie}(\mathrm{GL}_{m|n})$ for k a field. Consider the morphism

$$\mathrm{GL}_{m|n}(p)\colon\ \mathrm{GL}_{m|n}(A(\epsilon)) \to \mathrm{GL}_{m|n}(A), \quad \begin{pmatrix} p+\epsilon p' & q+\epsilon q' \\ r+\epsilon r' & s+\epsilon s' \end{pmatrix} \mapsto \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

with p, p', s, s' having entries in A_0 and q, q', r, r' having entries in A_1; the blocks p and s are invertible matrices. One can see immediately that

$$\mathrm{Lie}(\mathrm{GL}_{m|n})(A) = \ker(\mathrm{GL}_{m|n}(p)) = \left\{ \begin{pmatrix} I_m+\epsilon p' & \epsilon q' \\ \epsilon r' & I_n+\epsilon s' \end{pmatrix} \right\},$$

where I_n is an $n \times n$ identity matrix. The functor $\mathrm{Lie}(\mathrm{GL}_{m|n})$ is clearly group-valued and can be identified with the (additive) group functor $\mathrm{M}_{m|n}$ defined as (see Example 11.2.7)

$$\mathrm{M}_{m|n}(A) = \mathrm{Hom}_{(\mathrm{smod})}(\mathrm{M}(m|n)^*, A) = \mathrm{Hom}_{(\mathrm{salg})}(\mathrm{Sym}(\mathrm{M}(m|n)^*), A),$$

where $\mathrm{M}(m|n)$ is the super vector space

$$\mathrm{M}(m|n) = \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right\} \cong k^{m^2+n^2|2mn}$$

with P, Q, R, S respectively $(m \times m)$, $(m \times n)$, $(n \times m)$, $(n \times n)$-matrices with entries in k. An element $X \in \mathrm{M}(m|n)$ is even if $Q = R = 0$ and is odd if $P = S = 0$.

As we already noticed in Example 11.2.7, $\mathrm{M}(m|n)$ is a Lie superalgebra with superbracket

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX.$$

So $\mathrm{Lie}(\mathrm{GL}_{m|n})$ is a Lie superalgebra. In the next section we will see that in general we can give a Lie superalgebra structure to $\mathrm{Lie}(G)$ for any group scheme G.

(2) *The special linear Lie superalgebra.* A similar computation shows that

$$\mathrm{Lie}(\mathrm{SL}_{m|n})(A) = \left\{ W = \begin{pmatrix} I_m+\epsilon p' & \epsilon q' \\ \epsilon r' & I_n+\epsilon s' \end{pmatrix} \mid \mathrm{Ber}(W) = 1 \right\}.$$

The condition on the Berezinian is equivalent to

$$\det(I_n - \epsilon s') \det(I_m + \epsilon p') = 1,$$

which gives

$$\mathrm{tr}(p') - \mathrm{tr}(s') = 0.$$

Hence

$$\mathrm{Lie}(\mathrm{SL}_{m|n})(A) = \{X \in \mathrm{M}_{m|n}(A) \mid \mathrm{str}(X) = 0\},$$

where str is the supertrace, i.e., $\mathrm{str}\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$.

In the previous examples we have discovered that $\mathrm{Lie}(G)$ comes with a built-in $\mathcal{O}_k$-module structure. This is actually true in general, that is, for all superalgebras A, we have that $\mathrm{Lie}(G)(A)$ is an A_0-module in a functorial manner. In fact, let $u_a \colon A(\epsilon) \to A(\epsilon)$ be the endomorphism, $u_a(1) = 1$, $u_a(\epsilon) = a\epsilon$, for $a \in A_0$. u_a induces $G(u_a) \colon G(A(\epsilon)) \to G(A(\epsilon))$, that by compatibility conditions gives a well-defined morphism $\mathrm{Lie}(G)(u_a) \colon \mathrm{Lie}(G)(A) \to \mathrm{Lie}(G)(A)$. Hence there is a natural transformation $\mathcal{O}_k \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ such that

$$(a, x) \mapsto ax := \mathrm{Lie}(G)(u_a)x, \quad a \in \mathcal{O}_k(A),\ x \in \mathrm{Lie}(G)(A),$$

for any superalgebra A. One can immediately check for $\mathrm{GL}_{m|n}(A)$ and its subgroups that the morphism $(a, x) \mapsto ax$ corresponds to the multiplication of all the entries of the matrix x by $a \in A_0$.

We summarize our discussion with a proposition, the proof of which is a simple check that we leave to the reader.

Proposition 11.3.4. *Let G be a supergroup functor. Then* $\mathrm{Lie}(G)$ *is an $\mathcal{O}_k$-module with respect to the morphism $(a, x) \mapsto ax$ detailed above. That is:*

(1) $a(xy) = (ax)(ay)$,

(2) $(ab)x = a(bx)$ *for all* $a, b \in A_0$, $x, y \in \mathrm{Lie}(G)(A)$.

Moreover if $f : G \to H$ *is a morphism of supergroup functors, then* $\mathrm{Lie}(f)(ax) = a\,\mathrm{Lie}(f)(x)$, *that is, a morphism of supergroup functors induces a morphism* $\mathrm{Lie}(f)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *of* $\mathcal{O}_k$*-modules.*

We are now going to introduce the adjoint morphism, which plays an important role in the theory.

Definition 11.3.5. Let G be a supergroup functor. We define the *adjoint morphism* Ad as the natural transformation $\mathrm{Ad}\colon G \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ given by

$$G(A) \times \mathrm{Lie}(G)(A) \to \mathrm{Lie}(G)(A), \quad (g, x) \mapsto G(i)(g)xG(i)(g)^{-1}.$$

Notice that since $\mathrm{Lie}(G)(A) = \ker(G(p))$ is a normal subgroup in $G(A(\epsilon))$, we have $G(i)(g)xG(i)(g)^{-1} \in \mathrm{Lie}(G)(A) \subset G(A(\epsilon))$.

The following proposition establishes the naturality of our definition of Ad; its proof, amounting to trivial checks, is left to the reader.

Proposition 11.3.6. *The adjoint morphism* Ad *is compatible with the* $\mathcal{O}_k$*-module structure of* $\mathrm{Lie}(G)$*, in other words:*

(1) $\mathrm{Ad}(g)(xy) = \mathrm{Ad}(g)(x)\,\mathrm{Ad}(g)(y)$,

(2) $\mathrm{Ad}(g)(ax) = a\,\mathrm{Ad}(g)(x)$ *for all* $a, b \in A_0$, $x, y \in \mathrm{Lie}(G)(A)$.

11.4 Lie(G) for a supergroup scheme G

Let us now assume that G is a supergroup scheme over a field k.

We now want to show that $\mathrm{Lie}(G)\colon (\mathrm{salg}) \to (\mathrm{sets})$ is a representable functor and its representing superscheme is identified with the tangent space at the identity $T_{1_G}G$ of the supergroup G.

We start by showing that $\mathrm{Lie}(G)$ is isomorphic as an $\mathcal{O}_k$-module (i.e., as a super vector space) to $T_{1_G}G$, the tangent space to G at 1_G. Before this we need some general preliminaries.

Definition 11.4.1. Let $X = (|X|, \mathcal{O}_X)$ be a superscheme, $x \in |X|$. We define *the first neighbourhood of* X at a point $x \in |X|$, X_x, to be the superscheme $\underline{\mathrm{Spec}}\, \mathcal{O}_{X,x}/m_{X,x}^2$, where as usual $m_{X,x}$ is the maximal ideal in the local superring $\mathcal{O}_{X,x}$. The topological space $|X_x|$ consists of the one point $m_{X,x}$ which is the maximal ideal in $\mathcal{O}_{X,x}$.

Observation 11.4.2. There exists a natural morphism $f\colon X_x \to X$, from the first neighbourhood of X at x to the superscheme X. In fact we can write immediately the topological space map

$$|f|\colon \mathrm{Spec}\, \mathcal{O}_{X,x}/m_{X,x}^2 = \{m_{X,x}\} \to |X|, \quad m_{X,x} \mapsto x,$$

and the sheaf morphism

$$f_U^*\colon \mathcal{O}_X(U) \to \mathcal{O}_{X_x} \to \mathcal{O}_{X,x}/m_{X,x}^2 = \mathcal{O}_{X_x}(|f|^{-1}(U)),$$

where f_U^* is the composition of a natural map from $\mathcal{O}_X(U)$ to the direct limit $\mathcal{O}_{X,x}$ and the projection $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_{X,x}^2$ (with $x \in U$).

We now want to make some observations on the unit or identity element of a supergroup G. By definition we have that the identity is a superscheme morphism $e\colon \underline{\mathrm{Spec}}\, k \to G$. This corresponds to a natural transformation of the functor of points: $h_{\underline{\mathrm{Spec}}\, k} \to h_G$ assigning to the only morphism $1_A \in h_{\underline{\mathrm{Spec}}\, k}(A)$ a morphism that we will denote by $1_{G(A)} \in h_G(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, G)$. The topological space map $|1_{G(A)}|$ sends all the maximal ideals in Spec A to $1_G \in |G|$. The sheaf morphism $\mathcal{O}_G \to k$ is the evaluation at 1_G that is $\mathcal{O}_G(U) \to \mathcal{O}_{G,1_G} \to \mathcal{O}_{G,1_G}/m_{G,1_G} \cong k$ (the identity is a rational point by Observation 11.1.4). Hence it is immediate to verify that $1_{G(A)}$ *factors through* G_1 (the first neighbourhood at the identity 1_G), i.e.,

$$1_{G(A)}\colon \underline{\mathrm{Spec}}\, A \to G_1 = \underline{\mathrm{Spec}}\, \mathcal{O}_{G,1_G}/m_{G,1_G}^2 \to G.$$

This fact will be crucial in the proof of the next theorem.

Theorem 11.4.3. *Let G be a supergroup scheme. Then*

$$\mathrm{Lie}(G)(A) \cong \mathrm{Hom}_{(\mathrm{smod})}(m_{G,1_G}/m_{G,1_G}^2, A)$$

as $\mathcal{O}_k$-modules.

Proof. We start by defining the $\mathcal{O}_k$-linear natural transformation

$$\psi\colon\ \mathrm{Hom}_{(\mathrm{smod})}(m_{G,1_G}/m_{G,1_G}^2, A) \to \mathrm{Lie}(G)(A)$$

and then we will show that ψ is invertible by providing an explicit inverse.

Let $d : m_{G,1_G}/m_{G,1_G}^2 \to A$ be a linear map. Let d' be the linear map

$$d'\colon \mathcal{O}_{G,1_G}/m_{G,1_G}^2 \cong k \oplus m_{G,1_G}/m_{G,1_G}^2 \to A(\epsilon), \quad (s,t) \mapsto s + d(t)\epsilon.$$

So we have $d' \in h_{G_1}(A(\epsilon))$ since G_1 is the superscheme represented by $\mathcal{O}_{G,1_G}/m_{G,1_G}^2$.

This shows that we have a correspondence between $h_{G_1}(A)$ and the elements of $\mathrm{Hom}_{(\mathrm{smod})}(m_{G,1_G}/m_{G,1_G}^2, A)$. Let $\phi\colon G_1 \to G$ be the morphism described in Observation 11.4.2. By Yoneda's lemma, ϕ induces $\phi_{A(\epsilon)}\colon h_{G_1}(A(\epsilon)) \to h_G(A(\epsilon))$, hence we have an A_0-linear map

$$\psi\colon\ \mathrm{Hom}_{(\mathrm{smod})}(m_{G,1_G}/m_{G,1_G}^2, A) \to h_G(A(\epsilon)), \quad d \mapsto \phi_{A(\epsilon)}(d').$$

The following commutative diagram shows that $\psi(d) \in \ker(h_G(p)) = \mathrm{Lie}(G)(A)$:

$$\begin{array}{ccc} h_{G_1}(A(\epsilon)) & \xrightarrow{h_{G_1}(p)} & h_{G_1}(A) \\ d' & \longmapsto & 1_{G(A)} \\ \Big\downarrow{\scriptstyle \phi_{A(\epsilon)}} & & \Big\downarrow{\scriptstyle \phi_A} \\ h_G(A(\epsilon)) & \xrightarrow{h_G(p)} & h_G(A) \\ \psi(d) & \longmapsto & 1_{G(A)}. \end{array}$$

We now want to build an inverse for ψ. Let $z \in \ker(h_G(p)) \subset h_G(A(\epsilon))$, i.e., $h_G(p)z = 1_{G(A)}$, where

$$h_G(p)\colon h_G(A(\epsilon)) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A(\epsilon), G) \to h_G(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, G),\ z \mapsto 1_{G(A)}.$$

We have the commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Spec}}\, A(\epsilon) & \xrightarrow{z} & G \\ \Big\uparrow{\scriptstyle \underline{\mathrm{Spec}}\, p} & \nearrow{\scriptstyle 1_{G(A)}} & \\ \underline{\mathrm{Spec}}\, A, & & \end{array}$$

where $\underline{\mathrm{Spec}}\, p$ is the morphism induced by $p\colon A(\epsilon) \to A$.

Here z factors via G_1 because $1_{G(A)}$ splits via G_1. Since z factors via G_1, that is, $z\colon \underline{\mathrm{Spec}}\, A(\epsilon) \to G_1 \to G$, this provides immediately a map $\underline{\mathrm{Spec}}\, A(\epsilon) \to G_1$, i.e., an element in $h_{G_1}(A(\epsilon))$, corresponding to an element in $\mathrm{Hom}_{(\mathrm{smod})}(m_{G,1_G}/m^2_{G,1_G}, A)$.

So we have defined a map $\mathrm{Lie}(G)(A) \to \mathrm{Hom}_{(\mathrm{smod})}(m_{G,1_G}/m^2_{G,1_G}, A)$. One can check it is functorial and $\mathcal{O}_k$-linear. We leave to the reader the check that this is the inverse of ψ. □

Corollary 11.4.4. *Let G be a supergroup scheme. Then*

$$\mathrm{Lie}(G) \cong T_{1_G} G,$$

that is, $\mathrm{Lie}(G)$ *is identified as a super vector space with the tangent space to G at the identity.*

Proof. Immediate from Theorem 11.4.3 and Proposition 10.6.8. □

11.5 The Lie superalgebra of a supergroup scheme

We now want to show that $\mathrm{Lie}(G)$ is a Lie superalgebra for any supergroup scheme G over a field k. In other words we want to show that the functor $\mathrm{Lie}(G)\colon (\mathrm{salg}) \to (\mathrm{sets})$, defined in the previous section, is Lie algebra-valued and is represented by a super affine space, which carries a Lie superalgebra structure.

Our treatment of these topics again follows closely the classical discussion that is found in [23], Ch. II. We nevertheless find it necessary to repeat all of the arguments, given our different scope and in order to make the text self-contained.

We now want to define a natural transformation $[\ ,\ ]\colon \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ which has the properties of a bracket.

Let $\mathrm{GL}(\mathrm{Lie}(G))(A)$ be the (multiplicative) group of linear automorphisms of the super vector space $\mathrm{Lie}(G)$ and let $\mathrm{End}(\mathrm{Lie}(G))(A)$ be the (additive) group of linear endomorphisms of $\mathrm{Lie}(G)(A)$:

$$\mathrm{GL}(\mathrm{Lie}(G))(A) = \{\phi\colon \mathrm{Lie}(G)(A) \to \mathrm{Lie}(G)(A) \mid \phi \text{ invertible}\} \subset \mathrm{End}(\mathrm{Lie}(G))(A).$$

The natural $\mathcal{O}_k$-module structure of $\mathrm{Lie}(G)$ makes the two supergroup functors (one multiplicative the other additive)

$$\mathrm{GL}(\mathrm{Lie}(G))\colon (\text{salg}) \to (\text{sets}), \quad \mathrm{End}(\mathrm{Lie}(G))\colon (\text{salg}) \to (\text{sets})$$

valued in the A_0-linear morphisms of $\mathrm{Lie}(G)(A)$.

As in the ordinary setting, we have an identification between the Lie superalgebra of the general linear supergroup $\mathrm{Lie}(\mathrm{GL}(\mathrm{Lie}(G)))$ and the Lie superalgebra of endomorphisms $\mathrm{End}(\mathrm{Lie}(G))$, as we shall see in the next proposition.

Proposition 11.5.1. *Let V be a super vector space. Then*

$$\mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}(V).$$

Proof. Let us first define a natural transformation $\psi\colon \mathrm{End}(V) \to \mathrm{Lie}(\mathrm{GL}(V))$ and then we show that it is an isomorphism. Let $f \in \mathrm{End}(V)$. If ϵ is an even indeterminate $\epsilon^2 = 0$, we have $1 + \epsilon f \in \mathrm{End}(V)(A(\epsilon))$ and actually $1 + \epsilon f \in \mathrm{GL}(V)(A(\epsilon))$ since it is invertible, its inverse being $1 - \epsilon f$. Clearly $1 + \epsilon f \in \ker \mathrm{GL}(p) = \mathrm{Lie}(\mathrm{GL}(V))(A)$, so we define $\psi(f) = 1 + \epsilon f$. We have immediately that ψ is functorial and A_0-linear. Now we show that ψ is an isomorphism. Recall in general that if V is a super vector space we have

$$\mathrm{End}(V)(A(\epsilon)) := \mathrm{End}(V(A(\epsilon))) \cong \mathrm{End}(V(A)) \otimes A(\epsilon) = \mathrm{End}(V)(A) \otimes A(\epsilon).$$

Hence

$$\begin{aligned} \mathrm{End}(V)(A(\epsilon)) &\cong \mathrm{End}(V)(A) \oplus \epsilon\, \mathrm{End}(V)(A), \\ \mathrm{GL}(V)(A(\epsilon)) &\cong \mathrm{GL}(V)(A) \oplus \epsilon\, \mathrm{End}(V)(A), \end{aligned}$$

which gives us the result. □

As in the ordinary Lie theory, we can define the adjoint morphisms Ad and ad, which again play a key role. Though we have already defined Ad for supergroup functors, we prefer to repeat the definition in the context of supergroup schemes, since the notation will be different.

Definition 11.5.2. The *adjoint action* Ad of G on $\text{Lie}(G)$ is defined as the natural transformation

$$\text{Ad}\colon h_G \to \text{GL}(\text{Lie}(G)),$$
$$\text{Ad}(g)(x) = h_G(i)(g)xh_G(i)(g)^{-1}, \quad g \in h_G(A),\ x \in \text{Lie}(G)(A).$$

Notice that $\text{Ad}(g)x \in h_G(A(\epsilon))$; however, since $\ker(p)$ is a normal subgroup, we have $\text{Ad}(g)x \in \text{Lie}(G)(A)$.

The *adjoint action* ad of $\text{Lie}(G)$ on $\text{Lie}(G)$ is defined as

$$\text{ad} := \text{Lie}(\text{Ad})\colon\ \text{Lie}(G) \to \text{Lie}(\text{GL}(\text{Lie}(G))) = \text{End}(\text{Lie}(G)).$$

We are ready to define a *bracket* on $\text{Lie}(G)$ by

$$[x, y] := \text{ad}(x)y, \quad x, y \in \text{Lie}(G)(A).$$

Our goal is to prove that [,] is a Lie bracket for all superalgebras A in a functorial manner or, equivalently, that $\text{Lie}(G)$ is a Lie algebra-valued functor.

Observation 11.5.3. Let $c(g)\colon h_G(A) \to h_G(A)$ be the conjugation by an element $g \in |G|$: $c(g)(x) = gxg^{-1}$. Then

$$\text{Ad}(g) = \text{Lie}(c(g)).$$

This equality amounts to the commutativity of the diagram

$$\begin{array}{ccc}
\text{Lie}(G)(A) & \xrightarrow{\text{Lie}(c(g))} & \text{Lie}(G)(A) \\
\downarrow & & \downarrow \\
h_G(A(\epsilon)) & \xrightarrow{c(g)_{A(\epsilon)}} & h_G(A(\epsilon)) \\
\uparrow & & \uparrow \\
h_G(A) & \xrightarrow{c(g)_A} & h_G(A),
\end{array}$$

which is granted by the functoriality of our constructions.

We want to examine in detail the case in which $V = k^{m|n}$ so that $\text{GL}(V) = \text{GL}_{m|n}$.

Example 11.5.4. We want to see that in the case of $\text{GL}_{m|n}$ the Lie bracket $[x, y] := \text{ad}(x)y$ coincides with the bracket defined in Example 11.3.3 (1). We have

$$\text{Ad}\colon\ \text{GL}_{m|n}(A) \to \text{GL}(\text{Lie}(\text{GL}_{m|n}))(A) = \text{GL}(\text{M}_{m|n}(A)), \quad g \mapsto \text{Ad}(g).$$

Since $h_{\text{GL}_{m|n}}(i)\colon\ \text{GL}_{m|n}(A) \to \text{GL}_{m|n}(A(\epsilon))$ is an inclusion if we identify $\text{GL}_{m|n}(A)$ with its image, we can write

$$\text{Ad}(g)x = gxg^{-1}, \quad x \in \text{M}_{m|n}(A).$$

By definition we have $\mathrm{Lie}(\mathrm{GL}(\mathrm{M}_{m|n}))(A) = \{1 + \epsilon\beta \mid \beta \in \mathrm{End}(\mathrm{M}_{m|n})(A)\}$. Thus, for $a, b \in \mathrm{M}_{m|n}(A) \cong \mathrm{Lie}(\mathrm{GL}_{m|n})(A) = \{1 + \epsilon a \mid a \in \mathrm{M}_{m|n}(A)\}$ we have

$$\mathrm{ad}(1 + \epsilon a)b = (1 + \epsilon a)b(1 - \epsilon a) = b + (ab - ba)\epsilon = b + \epsilon[a, b].$$

Therefore, $\mathrm{ad}(1 + \epsilon a) = \mathrm{id} + \epsilon\beta(a)$, with $\beta(a) = [a,\ ]$.

It is important to observe that in $G(A(\epsilon))$ it is customary to write the product of two elements x and y as xy. However as elements of $\mathrm{Lie}(G)(A)$, their product is written as $x + y$ (hence the unit is 0 and the inverse of x is $-x$). In order to be able to switch between these two ways of writing, it is useful to introduce the notation $e^{\epsilon x}$. This is mainly a notational device and should not be interpreted as the exponential morphism in a strict sense.

Definition 11.5.5. Let $\phi\colon A(\epsilon) \to B(\alpha)$ be a superalgebra morphism such that $\phi(\epsilon) = \alpha$ and ϵ, α are two even indeterminates with square zero. Let $x \in \mathrm{Lie}(G)(A) \subset h_G(A(\epsilon))$. Define $e^{\alpha x} = h_G(\phi)(x) \in h_G(B(\alpha))$.

We have the following properties:

(1) $e^{\epsilon x} = x$ for all $x \in \mathrm{Lie}(G)(A)$. In fact, $e^{\epsilon x} = h_G(\mathrm{id})(x) = x$.

(2) $e^{\alpha(x+y)} = e^{\alpha x}e^{\alpha y}$. In fact, $e^{\alpha(x+y)} = h_G(\phi)(xy) = h_G(\phi)(x)h_G(\phi)(y) = e^{\alpha x}e^{\alpha y}$.

(3) $e^{\epsilon(ax)} = e^{a\epsilon x}$ for all $a \in A_0$.

Proposition 11.5.6. *Let G and H be two supergroups and let $f : h_G \to h_H$ be a natural transformation. Then*

$$f(e^{\alpha x}) = e^{\alpha\,\mathrm{Lie}(f)x}.$$

Proof. This is immediate from the commutativity of the diagram:

$$\begin{array}{ccc}
h_G(A(\epsilon)) \supset \mathrm{Lie}(G)(A) & \xrightarrow{\mathrm{Lie}(f)} & \mathrm{Lie}(H)(A) \subset h_H(A(\epsilon)) \\
x & \longmapsto & \mathrm{Lie}(f)(x) \\
\downarrow{\scriptstyle h_G(\phi)} & & \downarrow{\scriptstyle h_H(\phi)} \\
h_G(B(\alpha)) & \xrightarrow{f_{B(\alpha)}} & h_H(B(\alpha)) \\
e^{\alpha x} & \longmapsto & e^{\alpha\,\mathrm{Lie}(f)x}.
\end{array}$$

□

Remark 11.5.7. One can readily check that if $G = \mathrm{GL}(V)$, $\mathrm{Lie}(G) = \mathrm{End}(V)$, we have

$$e^{\epsilon x} = 1 + \epsilon x.$$

Proposition 11.5.8. *Let the notation be as above. Then*

$$\mathrm{Ad}(e^{\epsilon x})y = y + \epsilon[x, y] = (\mathrm{id} + \epsilon\, \mathrm{ad}(x))y \quad \textit{for all } x, y \in \mathrm{Lie}(G)(A).$$

Proof. By Proposition 11.5.6, we have

$$\mathrm{Ad}(e^{\epsilon x}) = e^{\epsilon\, \mathrm{Lie(Ad)}x} = e^{\epsilon\, \mathrm{ad}(x)} = \mathrm{id} + \epsilon\, \mathrm{ad}(x) \in \mathrm{GL}(\mathrm{Lie}(G))(A). \qquad \square$$

Lemma 11.5.9. *Let the notation be as above and let ϵ, ϵ' be two even elements with square zero. Then*

$$e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} e^{-\epsilon' y} = e^{\epsilon\epsilon'[x,y]} \in h_G(A(\epsilon, \epsilon')).$$

Proof. Reasoning as in Proposition 11.5.6 we have

$$e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} = e^{\epsilon' \mathrm{Ad}(e^{\epsilon x})y} = e^{\epsilon'(y + \epsilon[x,y])}$$

by Proposition 11.5.8. Therefore,

$$e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} = e^{\epsilon' y} e^{\epsilon\epsilon'[x,y]} = e^{\epsilon\epsilon'[x,y]} e^{\epsilon' y},$$

which gives the result. $\square$

Proposition 11.5.10. *The bracket* [,] *is antisymmetric.*

Proof. In the proof of the previous proposition we saw that

$$e^{\epsilon\epsilon'[x,y]} = e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} e^{-\epsilon' y} = e^{-\epsilon' y} e^{\epsilon x} e^{+\epsilon' y} e^{-\epsilon x} = e^{\epsilon\epsilon'[-y,x]}.$$

The result now follows from the next lemma. $\square$

Lemma 11.5.11. *If $e^{\epsilon\epsilon' u} = e^{\epsilon\epsilon' v}$, then $u = v$.*

Proof. Clearly if $e^{\epsilon u} = e^{\epsilon v}$, then $u = v$. Let us consider $\phi\colon A(\epsilon) \to A(\epsilon\epsilon')$, $\phi(\epsilon) = \epsilon\epsilon'$. ϕ induces $\psi = \underline{\mathrm{Spec}}\,\phi\colon \underline{\mathrm{Spec}}\,A(\epsilon\epsilon') \to \underline{\mathrm{Spec}}\,A(\epsilon)$. Clearly $|\psi|$ is a homeomorphism of the topological spaces since the spectrum of a ring is not influenced by the presence of the nilpotents. Moreover ψ^* is a monomorphism of sheaves. We claim that $h_X(\phi)\colon h_X(A(\epsilon)) \to h_X(A(\epsilon\epsilon'))$, $h_X(\phi)\alpha = \alpha \cdot \psi$, is one-to-one for all superschemes X, that is, $\alpha \cdot \psi = \beta \cdot \psi$ implies that $\alpha = \beta$. The fact $|\alpha| = |\beta|$ is clear. The statement $\alpha^* = \beta^*$ follows from the fact that we have a monomorphism of sheaves. $\square$

Proposition 11.5.12. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a morphism of supergroup functors. Then*

$$\mathrm{Lie}(\rho)[x, y] = [\mathrm{Lie}(\rho)(x), \mathrm{Lie}(\rho)(y)].$$

Proof. By Proposition s 11.5.6 and 11.5.8 we have

$$\rho(e^{\epsilon x}) = e^{\epsilon \operatorname{Lie}(\rho)x} = \mathrm{id} + \epsilon \operatorname{Lie}(\rho)x.$$

Using Lemma 11.5.9 we have

$$\rho(e^{\epsilon\epsilon'[x,y]}) = \rho(e^{\epsilon x})\rho(e^{\epsilon' y})\rho(e^{-\epsilon x})\rho(e^{-\epsilon' y}).$$

Hence

$$\begin{aligned}&\mathrm{id} + \epsilon\epsilon' \operatorname{Lie}(\rho)[x, y]\\ &\quad = (\mathrm{id} + \epsilon \operatorname{Lie}(\rho)x)(\mathrm{id} + \epsilon' \operatorname{Lie}(\rho)y)(\mathrm{id} - \epsilon \operatorname{Lie}(\rho)x)(\mathrm{id} - \epsilon' \operatorname{Lie}(\rho)y),\end{aligned}$$

which immediately gives

$$\operatorname{Lie}(\rho)[x, y] = [\operatorname{Lie}(\rho)(x), \operatorname{Lie}(\rho)(y)]. \qquad \square$$

Proposition 11.5.13. *The bracket* [,] *satisfies the Jacobi identity.*

Proof. In the previous proposition take $\rho = \mathrm{Ad}$. Then we have

$$[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}([x, y]) \quad \text{for all } x, y \in \operatorname{Lie}(G)(A),$$

which gives us

$$\begin{aligned}\mathrm{ad}(x)\,\mathrm{ad}(y)z - \mathrm{ad}(y)\,\mathrm{ad}(x)z &= \mathrm{ad}([x, y])z,\\ [x[y, z]] - [y, [x, z]] &= [[x, y], z],\end{aligned}$$

which is the Jacobi identity. □

Corollary 11.5.14. *The natural transformation* [,]: $\operatorname{Lie}(G) \times \operatorname{Lie}(G) \to \operatorname{Lie}(G)$ *defined as*

$$[x, y] := \mathrm{ad}(x)y, \quad x, y \in \operatorname{Lie}(G)(A),$$

is a Lie bracket for all A. $\operatorname{Lie}(G)$ *is a Lie algebra functor and is represented by a Lie superalgebra (still denoted by* $\operatorname{Lie}(G)$*).*

Proof. Immediate from previous propositions. □

Corollary 11.5.15. *If* $\rho: G \to \mathrm{GL}(V)$ *is a morphism of supergroups, then* $\operatorname{Lie}(\rho)$: $\operatorname{Lie}(G) \to \operatorname{Lie}(\mathrm{GL}(V)) \cong \operatorname{End}(V)$ *is a morphism of Lie algebra-valued functors and corresponds to a Lie superalgebra morphism of their representing Lie superalgebras.*

11.6 Affine algebraic supergroups

Let k be a field.

We now want to restrict our attention and assume that the supergroup scheme G is an affine algebraic supergroup. In this case our supergroup G can be effectively replaced by its (finitely generated) Hopf superalgebra that we shall denote by $k[G]$. Let Δ, ϵ, S denote respectively the comultiplication, counit and antipode of $k[G]$.

Definition 11.6.1. We define the additive map $D\colon k[G] \to k[G]$ to be a *super derivation* of $k[G]$ if it satisfies the following properties:

(1) D is k-linear, i.e., $D(a) = 0$ for all $a \in k$.

(2) D satisfies the Leibniz identity, $D(fg) = D(f)g + (-1)^{p(D)p(f)} f D(g)$.
D is called *left-invariant* if further:

(3) $\Delta \circ D = (\mathrm{id} \otimes D) \circ \Delta$, where Δ denotes the comultiplication in $k[G]$.

Similarly we say that the additive map $d\colon k[G] \to k$ is a *superderivation* if it is k-linear and satisfies the Leibniz identity

$$d(fg) = d(f)\epsilon(g) + (-1)^{p(d)p(f)}\epsilon(f)d(g).$$

Notice that in the definition of superderivation $d\colon k[G] \to k$ the sign $(-1)^{p(d)p(f)}$ can be omitted: in fact, whenever $p(f) = 1$, we have $\epsilon(f) = 0$ since ϵ is a morphism and therefore it preserves the parity.

We denote by $\mathrm{Der}(k[G], k[G])$ the set of superderivations $D\colon k[G] \to k[G]$ and b $\mathrm{Der}(k[G], k)$ the set of superderivations $d\colon k[G] \to k$.

Proposition 11.6.2. *The set $L(G)$ of left-invariant derivations of $k[G]$ is a Lie superalgebra with bracket*

$$[D_1, D_2] := D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1.$$

Proof. The fact that [,] is a superbracket is a straightforward calculation. We only want to check that for all D_1, D_2 left-invariant derivations, $[D_1, D_2]$ is a left-invariant derivation, i.e., it satisfies the properties (1), (2), (3) as in Definition 11.6.1. (1) is clear, (2) is a tedious calculation that we leave to the reader as an exercise. As for (3), since D_1 and D_2 are left-invariant derivations, we can write

$$\begin{aligned}(\Delta \circ D_1 D_2) f &= (\Delta \circ D_1)(D_2 f)\\ &= (\mathrm{id} \otimes D_1 \circ \Delta \circ D_2)(f)\\ &= ((\mathrm{id} \otimes D_1) \circ (\mathrm{id} \otimes D_2) \circ \Delta)(f)\\ &= ((\mathrm{id} \otimes D_1 D_2) \circ \Delta)(f).\end{aligned}$$

Similarly,

$$\Delta \circ (-1)^{p(D_1)p(D_2)} D_2 D_1 = (\mathrm{id} \otimes (-1)^{p(D_1)p(D_2)} D_2 D_1) \circ \Delta.$$

Hence

$$\Delta \circ [D_1, D_2] = (\mathrm{id} \otimes [D_1, D_2]) \circ \Delta,$$

as we wanted to show. □

In Chapter 7 we have seen the identification between the left-invariant vector fields and the Lie superalgebra of a Lie supergroup. In the algebraic context the next theorem provides an analogy establishing a one-to-one correspondence between the left-invariant derivations and the Lie superalgebra of an affine algebraic supergroup.

Theorem 11.6.3. *Let G be an affine supergroup scheme. Then we have natural bijections among the sets:*

(1) *$L(G)$ the left-invariant derivations in* $\mathrm{Der}(k[G], k[G])$,

(2) $\mathrm{Der}(k[G], k)$,

(3) $\mathrm{Lie}(G)$.

Proof. Let us examine the correspondence between (1) and (2). We want to construct an invertible map $\phi\colon \mathrm{Der}(k[G], k) \to L(G)$. Let $d \in \mathrm{Der}(k[G], k)$. Define $\phi(d) = (\mathrm{id} \otimes d)\Delta$. Such $\phi(d)$ is left-invariant; in fact,

$$\begin{aligned}(\Delta \circ \phi(d))(f) &= (\Delta \circ \mathrm{id} \otimes d \circ \Delta)(f)\\ &= \Delta(\textstyle\sum f^{(1)} d(f^{(2)}))\\ &= (\mathrm{id} \otimes \mathrm{id} \otimes d)(\textstyle\sum \Delta(f^{(1)}) \otimes f^{(2)})\\ &= ((\mathrm{id} \otimes \mathrm{id} \otimes d) \circ (\Delta \otimes \mathrm{id}) \circ \Delta)(f),\end{aligned}$$

where we use the Sweedler notation: $\Delta(f) = \sum f^{(1)} \otimes f^{(2)}$. On the other hand we have

$$\begin{aligned}(\mathrm{id} \otimes \phi(d))\Delta(f) &= ((\mathrm{id} \otimes \mathrm{id} \otimes d) \circ (\mathrm{id} \otimes \Delta) \circ \Delta)(f)\\ &= (\mathrm{id} \otimes (\mathrm{id} \otimes d \circ \Delta) \circ \Delta)(f)\end{aligned}$$

which is the same as before since by the coassociativity axiom in a Hopf superalgebra $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

Moreover, $\phi(d)$ is a derivation. In fact, since d is a derivation, it follows that

$$\begin{aligned}\phi(d)(fg) &= (\mathrm{id} \otimes d) \circ \Delta(fg)\\ &= \textstyle\sum f^{(1)} g^{(1)} d(f^{(2)} g^{(2)})\\ &= \textstyle\sum f^{(1)} g^{(1)} (d(f^{(2)})\epsilon(g^{(2)}) + (-1)^{p(d)p(f)} \epsilon(f^{(2)}) d(g^{(2)})).\end{aligned}$$

We can rewrite the last expression as (m denotes the multiplication)

$$\begin{aligned}&m(\textstyle\sum (f^{(1)} \otimes d(f^{(2)}))(g^{(1)} \otimes \epsilon(g^{(2)})))\\ &\qquad + (-1)^{p(d)p(f)} m((f^{(1)} \otimes \epsilon(f^{(2)}))(g^{(1)} \otimes d(g^{(2)})))\\ &= (\mathrm{id} \otimes d)\Delta(f)(\mathrm{id} \otimes \epsilon)\Delta(g) + (-1)^{p(d)p(f)} (\mathrm{id} \otimes \epsilon)\Delta(f)(\mathrm{id} \otimes d)\Delta(g)\\ &= \phi(d)(f)g + (-1)^{p(d)p(f)} f\phi(d)(g),\end{aligned}$$

since we recall that, by the Hopf superalgebra axioms, $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$. Then $\phi(d) \in \mathrm{Der}(k[G], k[G])$. Vice versa, if $D \in L(G)$ define $\psi(D) = \epsilon \circ D$. We leave to the reader, as an easy exercise, the check that ψ is a derivation. One can check also that ψ is the inverse of ϕ. We now want a correspondence between (b) and (c). By Theorem 11.4.3 we have $\mathrm{Lie}(G) = \mathrm{Hom}_{(\mathrm{smod})}(T_{1_G}(G)^*, A) = \mathrm{Der}(\mathcal{O}_{G,1_G}, k)$. Observe that, as in the ordinary case,

$$\mathrm{Der}(\mathcal{O}_{G,1_G}, k) = \mathrm{Der}(k[G], k),$$

that is, the derivation on the localization of the ring $k[G]$ is determined by the derivation on the ring itself. □

11.7 Linear representations

We now want to discuss linear representations. In particular we will show that, as in the classical case, every affine algebraic supergroup G can be embedded into some $\mathrm{GL}_{m|n}$.

We start by recalling the notion of closed embedding discussed in the previous chapter.

Definition 11.7.1. Let $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ be two superschemes and let $f: X \to Y$ be a superscheme morphism. We say that f is a *closed embedding* if the topological map $|f|: |X| \to |Y|$ is a homeomorphism of the topological space $|X|$ onto its image in $|Y|$ and the sheaf map $f^*: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a surjective morphism of sheaves of superalgebras.

This means that we may identify X with a closed subscheme of Y, so its sheaf is identified with $\mathcal{O}_Y / \mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$. If both X and Y are affine superschemes we have by Remark 10.1.7 that f is a closed embedding if and only if $\mathcal{O}(X) \cong \mathcal{O}(Y)/I$ for some ideal I.

We now want to introduce the notion of linear representation of a supergroup. Let $h_G: (\mathrm{salg}) \to (\mathrm{sets})$, $h_G(A) = \mathrm{Hom}(k[G], A)$ be the functor of points of our affine algebraic supergroup G, with Hopf superalgebra $k[G]$.

Definition 11.7.2. Let V be a super vector space. We define *linear representation* of G in V to be a natural transformation ρ, which is a morphism of supergroup functors:

$$\rho: h_G \to \mathrm{GL}(V).$$

Here $\mathrm{GL}(V)$ is, as usual, the functor

$$\mathrm{GL}(V): (\mathrm{salg}) \to (\mathrm{sets}), \quad \mathrm{GL}(V)(A) = \mathrm{GL}(A \otimes V),$$

and $\mathrm{GL}(A \otimes V)$ denotes the automorphisms of the A-module $A \otimes V$ preserving the parity. We will also say that G *acts* on V.

Definition 11.7.3. Let V be a super vector space. Then V is said to be a *left G-comodule* if there exists a linear map

$$\Delta_V : V \to k[G] \otimes V,$$

called a *comodule map*, with the properties

(1) $(\mathrm{id}_G \otimes \Delta_V)\Delta_V = (\Delta \otimes \mathrm{id}_V)\Delta_V$,

(2) $(\epsilon \otimes \mathrm{id}_V)\Delta_V = \mathrm{id}_V$,

where $\mathrm{id}_G : k[G] \to k[G]$ is the identity map.

One can also define a right G-comodule in the obvious way.

Observation 11.7.4. The two notions of G acting on V and V being a (left) G-comodule are essentially equivalent. In fact, given a representation $\rho\colon G \to \mathrm{GL}(V)$, it defines immediately a comodule map

$$\Delta_V(v) = \rho_{k[G]}(\mathrm{id}_G)v, \quad \mathrm{id}_G \in h_G(k[G]) = \mathrm{Hom}_{(\mathrm{salg})}(k[G], k[G]),$$

where we are using the natural identification (for $A = k[G]$)

$$\mathrm{GL}(V)(A) \subset \mathrm{End}(V)(A) \cong \mathrm{Hom}_{(\mathrm{smod})}(V, A \otimes V).$$

Vice versa if we have a comodule map Δ_V we can define a representation

$$\rho_A : h_G(A) \to \mathrm{GL}(V)(A) \subset \mathrm{Hom}_{(\mathrm{smod})}(V, A \otimes V), \quad g \mapsto v \mapsto (g \otimes \mathrm{id})(\Delta_V(v)),$$

where $g \in h_G(A) = \mathrm{Hom}_{(\mathrm{salg})}(k[G], A)$.

Let us look at this correspondence in a special but important case.

Example 11.7.5. Let us consider the natural action of $\mathrm{GL}_{m|n}$ on $k^{m|n}$:

$$\rho_A\colon \mathrm{GL}_{m|n}(A) \to \mathrm{GL}(k^{m|n})(A), \quad g = (g_{ij}) \mapsto e_j \mapsto \sum g_{ij} \otimes e_i,$$

Here $\{e_j\}$ is the canonical homogeneous basis for the super vector space $k^{m|n}$. We identify the morphism $g \in \mathrm{GL}_{m|n}(A) = \mathrm{Hom}_{(\mathrm{salg})}(k[\mathrm{GL}_{m|n}], A)$ with the matrix with entries $g_{ij} = g(x_{ij}) \in A$, where x_{ij}'s are the generators of $k[\mathrm{GL}_{m|n}]$.

This corresponds to the comodule map

$$\Delta_{k^{m|n}} : k^{m|n} \to k[\mathrm{GL}_{m|n}] \otimes k^{m|n}, \quad e_j \mapsto \sum x_{ij} \otimes e_i.$$

Vice versa, the comodule map $e_j \mapsto \sum x_{ij} \otimes e_i$ corresponds to the representation

$$\begin{aligned}\rho_A\colon\ & \mathrm{GL}_{m|n}(A) \to \mathrm{GL}(k^{m|n})(A),\\ & g = (g_{ij}) \mapsto e_j \mapsto (g \otimes \mathrm{id})(\textstyle\sum x_{ij} \otimes e_i) = \sum g_{ij} \otimes e_i.\end{aligned}$$

Definition 11.7.6. Let G act on the superspace V via a representation ρ corresponding to the comodule map Δ_V. We say that the subspace $W \subset V$ is *G-stable* if $\Delta_V(W) \subset k[G] \otimes W$. Equivalently W is G-stable if $\rho_A(g)(A \otimes W) \subset A \otimes W$.

Definition 11.7.7. The *right regular representation* of the affine algebraic group G is the representation of G in the (infinite-dimensional) super vector space $k[G]$ corresponding to the comodule map

$$\Delta \colon k[G] \to k[G] \otimes k[G].$$

Proposition 11.7.8. *Let ρ be a linear representation of an affine algebraic supergroup G. Then each finite-dimensional subspace W of V is contained in a finite-dimensional G-stable subspace of V.*

Proof. It is the same as in the commutative case. Let us sketch it for the case of a right representation, the case of left being the same. It is enough to prove for W generated by one element $x \in V$. Let $\Delta_V : V \to V \otimes k[G]$ be the comodule structure associated to the representation ρ. Let

$$\Delta_V(x) = \sum_i x_i \otimes a_i$$

where $\{a_i\}$ is a basis for $k[G]$.

We claim that $\mathrm{span}_k\{x_i\}$ is a G-stable subspace.

By definition of comodule we have

$$(\Delta_V \otimes \mathrm{id}_G)(\Delta_V(x)) = (\mathrm{id}_V \otimes \Delta)(\Delta_V(x)),$$

that is,

$$\sum_j \Delta_V(x_j) \otimes a_j = \sum_j x_j \otimes \Delta(a_j) = \sum_{i,j} x_j \otimes b_{ij} \otimes a_i.$$

Hence

$$\Delta_V(x_i) = \sum_j x_j \otimes b_{ij}.$$

The finite-dimensional stable subspace is given by the span of the x_i's.

Moreover we have $x \in \mathrm{span}_k\{x_i\}$. In fact by property (2) of Definition 11.7.3 we have

$$(\mathrm{id} \otimes \epsilon)\Delta_V(x) = \mathrm{id}_V(x), \text{ that is, } x = \sum x_j \epsilon(b_{ij}). \qquad \square$$

Theorem 11.7.9. *Let G be an affine supergroup variety. Then there exists a closed embedding of algebraic supergroups*

$$G \subset \mathrm{GL}_{m|n}$$

for suitable m and n.

Proof. By Definition 11.7.1 we need to find a surjective superalgebra morphism $k[\mathrm{GL}_{m|n}] \to k[G]$ for suitable m and n. Let $k[G] = k[f_1, \dots, f_n]$, where f_i are homogeneous and chosen so that $W = \mathrm{span}\{f_1, \dots, f_n\}$ is G-stable, according to the right regular representation. This choice is possible because of Proposition 11.7.8. We have

$$\Delta_{k[G]}(f_i) = \sum_j f_j \otimes a_{ij}, \quad \Delta_{k[G]}(a_{ij}) = \sum_j a_{ik} \otimes a_{kj}.$$

Define the morphism

$$\phi\colon k[\mathrm{GL}_{m|n}] \to k[G], \quad x_{ij} \mapsto a_{ij},$$

where x_{ij} are the generators for $k[\mathrm{GL}_{m|n}]$. This is the required surjective superalgebra morphism. In fact, since $k[G]$ is both a right and left G-comodule, we have

$$f_i = (\epsilon \otimes \mathrm{id})\Delta(f_i) = (\epsilon \otimes \mathrm{id})(\textstyle\sum_j f_j \otimes a_{ij}) = \sum_j \epsilon(f_j)a_{ij},$$

which proves the surjectivity. This shows that we have an embedding of G into $\mathrm{GL}_{m|n}$ as superschemes. In order to see that this is also a supergroup morphism it is enough to check that ϕ is a Hopf superalgebra morphism, but this is a straightforward verification. □

We thus have proved the following.

Corollary 11.7.10. *G is an affine supergroup scheme if and only if it is a closed subgroup of $\mathrm{GL}_{m|n}$ for suitable m and n.*

Proposition 11.7.11. *Let the G be an affine algebraic supergroup and H a closed subsupergroup, i.e., $k[H] = k[G]/I$. Then there exists a finite-dimensional representation of G, with super vector space V and a subspace $W \subset V$, which is fixed by H.*

Proof. Let $\widehat{V} = k[G]$ and $\widehat{W} = I$. We have immediately defined two comodule maps $\Delta_{\widehat{V}} = \Delta_G$, $\Delta_{\widehat{W}} = \Delta_G|_I$ where Δ_G is the comultiplication in the Hopf algebra $k[G]$:

$$\Delta_{\widehat{V}}\colon \widehat{V} \to \widehat{V} \otimes k[G], \quad \Delta_{\widehat{W}}\colon \widehat{W} \to \widehat{W} \otimes k[H].$$

The second map is induced by the first one, in fact

$$\Delta_G(I) = I \otimes k[G] + k[G] \otimes I$$

since I is a Hopf ideal. It is simple to check from the definitions that such a comodule map corresponds to the following action of G on the super vector space $k[G]$:

$$(g \cdot f)(x) = f(xg), \quad g, x \in G(A), \ f \in k[G],$$

where customarily we denote $x(f)$ by $f(x)$ and as usual $xg = m \circ (x \otimes g) \circ \Delta$ (see Proposition 11.1.2).

By Proposition 11.7.8 we can find a finite-dimensional subspace V invariant under the action of G and containing all the generators of $k[G]$ and a subspace $W = I \cap V$ containing the generators of I, hence fixed by H. □

11.8 The algebraic stabilizer theorem

In this section we want to prove the stabilizer theorem, which asserts the representability of the stabilizer functor for the action of an affine supergroup on an affine supervariety. This is important since it allows us to prove that all the classical algebraic supergroups are representable and to compute explicitly their Lie superalgebras.

Definition 11.8.1. We say that an algebraic supergroup G *acts* on a superscheme X if we have a morphism $\rho\colon G \times X \to X$ corresponding to the functorial family of morphisms

$$\rho_A\colon h_G(A) \times h_X(A) \to h_X(A), \quad (g, x) \mapsto g \cdot x \quad \text{for all } g \in h_G(A),\ x \in h_X(A),$$

satisfying the following properties:

(1) $1 \cdot x = x$ for all $x \in h_X(A)$.

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in h_G(A)$ and $x \in h_X(A)$.

Let u be a rational topological point of X, that is, $u \in |X|$, or equivalently $u \in h_X(k) = \mathrm{Hom}(\mathcal{O}(X), k)$. Let m_u be the maximal ideal corresponding to u in $\mathcal{O}_{X,u}$. Notice that u can be viewed naturally as an A-point u_A for all superalgebras A since $k \subset A$. So we have a morphism

$$\tau\colon G \to X, \quad g \mapsto g \cdot u_A, \quad g \in h_G(A),\ u_A \in h_X(A).$$

If G and X are affine, we can write equivalently a morphism

$$\tilde{\tau}\colon \mathcal{O}(X) \to \mathcal{O}(G).$$

Definition 11.8.2. We call *stabilizer supergroup functor* of the point $u \in |X|$ with respect to the action ρ the group-valued functor $\mathrm{Stab}_u\colon (\mathrm{salg}) \to (\mathrm{sets})$ defined by

$$\mathrm{Stab}_u(A) = \{g \in h_G(A) \mid \tau_A(g) = g \cdot u_A = u_A\},$$

where $\tau_A\colon h_G(A) \to h_X(A)$, or equivalently if G and X are affine,

$$\mathrm{Stab}_u(A) = \{g \in h_G(A) = \mathrm{Hom}(\mathcal{O}(G), A) \mid g \circ \tilde{\tau} = u_A\}.$$

We want to prove that this functor is representable by an affine supergroup.

Theorem 11.8.3. *Let G be an affine supergroup acting on an affine supervariety X and let u be a topological rational point of X. Then* Stab_u *is an affine supergroup.*

Proof. The stabilizer can be described in an equivalent way as

$$\mathrm{Stab}_u(A) = \{g \in h_G(A) \mid (g \circ \tilde{\tau})|_{m_u} = 0\},$$

where $m_u \subset \mathcal{O}(X)$ is the ideal of the topological point u. Let I be the ideal in $\mathcal{O}(G)$ generated by $\tilde{\tau}(x)$ for all $x \in m_u$. One can immediately check that $g \in h_G(A) = \mathrm{Hom}(\mathcal{O}(G), A)$ is in $\mathrm{Stab}_u(A)$ if and only if g factors via $\mathcal{O}(G)/I$, that is, $g\colon \mathcal{O}(G) \to \mathcal{O}(G)/I \to A$. So we have $\mathrm{Stab}_u(A) = \mathrm{Hom}(\mathcal{O}(G)/I, A)$. □

We now want to describe some important applications of this result, namely, the representability of the supergroup functors corresponding to the classical Lie superalgebras (for a complete description of such superalgebras we refer to Appendix A).

1. *$A(m|n)$ series.* $A(m|n)$ consists of the matrices in the super vector space $\mathrm{M}(m+1|n+1)$ with supertrace zero. This is a Lie superalgebra with the bracket induced by the one in $\mathrm{M}(m+1|n+1)$.

Consider the morphism

$$\rho\colon \mathrm{GL}_{m|n} \times k^{1|0} \to k^{1|0}, \quad (g, c) \mapsto \mathrm{Ber}(g)c.$$

The multiplicative property of Ber ensures this is a well-defined action of $\mathrm{GL}_{m|n}$ on $k^{1|0}$. The stabilizer of the point $1 \in k^{1|0}$ coincides with all the matrices in $\mathrm{GL}_{m|n}(A)$ with Berezinian equal to 1, that is, the special linear supergroup $\mathrm{SL}_{m|n}(A)$. From Theorem 11.8.3 it follows immediately that $\mathrm{SL}_{m|n}$ is representable. Moreover in Example 11.3.3 we checked that $A(m|n) = \mathrm{Lie}(\mathrm{SL}_{m+1|n+1})$.

2. *$B(p|q)$, $C(p)$, $D(p|q)$ series.* Consider the morphism

$$\rho\colon \mathrm{GL}_{m|n} \times \mathcal{B} \to \mathcal{B}, \quad (g, \psi(\cdot,\cdot)) \mapsto \psi(g\cdot, g\cdot),$$

where $\mathcal{B}$ is the super vector space of all the symmetric bilinear forms on $k^{m|2n}$. Consider the point in $\mathcal{B}$:

$$\Phi = \begin{pmatrix} 0 & I_p & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_q \\ 0 & 0 & 0 & -I_q & 0 \end{pmatrix} \quad \text{if } m = 2p+1, n = 2q,$$

or

$$\Phi = \begin{pmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix} \quad \text{if } m = 2p, n = 2q.$$

We define the stabilizer of the point Φ to be the supergroup functor $\mathrm{Osp}_{m|2n}$. Again this is an algebraic supergroup by Theorem 11.8.3.

In order to compute its Lie superalgebra, let us consider a generic matrix $M = \begin{pmatrix} I+\epsilon X & \epsilon Y \\ \epsilon W & I+\epsilon Z \end{pmatrix}$ in $\mathrm{M}(m|n) \cong \mathrm{Lie}(\mathrm{GL}_{m|n})$. We are assuming $m = 2p$ even since for m odd the calculation is very much the same. The condition for M to belong to

$\text{Lie}(\text{Osp}_{2p|2q})$ is $M^t \Phi M = \Phi$, which is

$$\begin{pmatrix} I+\epsilon X^t & \epsilon W^t \\ \epsilon Y^t & I+\epsilon Z^t \end{pmatrix} \begin{pmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix} \begin{pmatrix} I+\epsilon X & \epsilon Y \\ \epsilon W & I+\epsilon Z \end{pmatrix}$$
$$= \begin{pmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix}.$$

After a tedious calculation one finds that the generic matrix M in $\text{Lie}(\text{Osp}_{2p|2q})$ has the form

$$\left\{ \begin{pmatrix} a & b & x & x_1 \\ c & -a^t & y & y_1 \\ y_1^t & x_1^t & d & e \\ -y^t & -x^t & f & -d \end{pmatrix} \right\},$$

where the matrices b and c are skewsymmetric and f and e are symmetric and the entries of all the matrices $a, b, c, d, e, f, x, x_1, y, y_1, u, v$ are in k. This shows that $\text{Lie}(\text{Osp}_{2p|2q}) = \text{osp}(2p|2q)$, in fact the matrices in $\text{osp}(2p|2q)$ have precisely the form prescribed above.

So we have

$$B(p|q) := \text{osp}(2p+1|2q) = \text{Lie}(\text{Osp}_{2p+1|2q}),$$
$$C(q) := \text{osp}(2|2q-2) = \text{Lie}(\text{Osp}_{2|2q-2})$$

and

$$D(p|q) := \text{osp}(2p|2q) = \text{Lie}(\text{Osp}_{2p|2q}).$$

3. $P(n)$ *series*. Define the algebraic supergroup $\pi S p_{n|n}$, as we did for $\text{Osp}_{m|n}$ by taking antisymmetric bilinear forms instead of symmetric ones. Consider the action

$$\pi \text{Sp}_{n|n} \times k^{1|0} \to k^{1|0}, \quad (g, c) \mapsto \text{Ber}(g)c.$$

By Theorem 11.8.3 we have that Stab_1 is an affine algebraic supergroup, hence it is a Lie supergroup. It corresponds to the $P(n)$ series of classical Lie superalgebras (see Appendix A).

4. $Q(n)$ *series*. Let $D = k[\eta]/(\eta^2+1)$. This is a noncommutative superalgebra. Define the supergroup functor $\text{GL}_n(D)\colon (\text{salg}) \to (\text{sets})$, with $\text{GL}_n(D)(A)$ the group of automorphisms of the left supermodule $A \otimes D$. In [22] is proven the existence of a morphism, called the *odd determinant*,

$$\text{odet}\colon \text{GL}_n(D) \to k^{0|1}.$$

Reasoning as before, define

$$\mathrm{GL}_n(D) \times k^{0|1} \to k^{0|1}, \quad (g, c) \mapsto \mathrm{odet}(g)c.$$

Then $G = \mathrm{Stab}_1$ is an affine algebraic supergroup, and for $n \geq 2$ we define $Qg(n)$ as the quotient of G and the diagonal subgroup $\mathrm{GL}_{1|0}$. This is an algebraic and Lie supergroup and its Lie superalgebra is $Q(n)$.

11.9 References

In [44] Kac proved a classification theorem for simple Lie superalgebras that we have summarized in Appendix A. The description of the supergroup functors, corresponding to the classical super series of Lie superalgebras introduced by Kac, appeared in [22], p. 70; however no representability statement of the supergroups was proved there. In [14], [15] all the classical supergroup functors together with the Hopf superalgebras representing them are described in detail.

A

Lie superalgebras

The purpose of this appendix is to give a brief introduction to the theory of Lie superalgebras and, in particular, to the theory of representations of classical Lie superalgebras. This material is well known and is mostly found in the works of Kac, [44], [45]. This appendix is self-contained and does not require knowledge of any material of the previous chapters. However, we assume basic knowledge of semisimple Lie algebras.

We start by describing the classification of simple finite-dimensional complex Lie superalgebras. We do not provide any details on the proof of the main Theorem A.1.7, but we give a full description of the classical families of simple Lie superalgebras, including the root systems, Cartan matrices, and Dynkin diagrams.

We then discuss the finite-dimensional representations of classical Lie superalgebras. In the ordinary setting, the finite-dimensional modules for the special linear Lie algebra sl_2 play a key role. Ultimately, for a Lie algebra $\mathfrak{g}$, the conditions to impose to have a finite-dimensional $\mathfrak{g}$-module, boil down to the conditions to impose in order to have finite-dimensionality for the sl_2-modules, corresponding to simple roots. The situation for classical Lie superalgebras is different. In the ordinary setting, the root vectors X_α and $X_{-\alpha}$, corresponding to a pair of opposite roots α and $-\alpha$, always generate an sl_2 subalgebra inside the given simple Lie algebra. In contrast, given a simple Lie superalgebra $\mathfrak{g}$ and a root α of $\mathfrak{g}$, the root vectors corresponding to roots proportional to α may generate a subalgebra of $\mathfrak{g}$ isomorphic to sl_2, $\mathrm{osp}(1|2)$, $\mathrm{sl}(1|1)$, $\mathrm{sq}(2)$, or even a nilpotent subalgebra. Hence in order to fully understand the representation theory of Lie superalgebras it is necessary to study in detail and classify all irreducible representations of all these Lie superalgebras. We shall describe the irreducible representations for sl_2, $\mathrm{osp}(1|2)$, $\mathrm{sl}(1|1)$, and then we will give the main theorem on the classification of finite-dimensional representations of basic classical Lie superalgebras, describing in detail the case of $\mathrm{osp}(2m+1|2n)$. This example is very illuminating since we can see how the extra conditions for the finite-dimensionality of the modules make their appearance once we take into account all Borel subalgebras at once. In the ordinary setting this is not necessary since if we fix a Borel subalgebra, all other Borel subalgebras are conjugate to the fixed one under the action of the Weyl group. For Lie superalgebras, the Weyl group, still a classical object, appears to be "too small" and not all Borel subalgebras are conjugate. For this reason, in order to proceed to the classification of finite-dimensional representations, we need to use the *odd reflections* to compensate for the lack of enough symmetries in the Weyl group. In Section A.5 we will see explicitly in the example of an orthosymplectic Lie superalgebra how this happens.

A.1 Classical Lie superalgebras

We assume the ground field k is algebraically closed and of characteristic zero. For concreteness the reader may think of k as $\mathbb{C}$.

We recall a few definitions from Chapter 1, since here our notation is slightly different to adhere to the conventions in representation theory of Lie superalgebras.

Throughout this appendix, by a *Lie superalgebra*, we mean a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bracket $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is a bilinear even map and satisfies the following properties:

(1) Anti-symmetry: $[x, y] + (-1)^{|x||y|}[y, x] = 0$ for homogeneous elements x, y in $\mathfrak{g}$.

(2) Jacobi identity:

$$[x, [y, z]] + (-1)^{|x||y|+|x||z|}[y, [z, x]] + (-1)^{|y||z|+|x||z|}[z, [x, y]] = 0$$

for homogeneous elements $x, y, z \in \mathfrak{g}$. $|\cdot|$ as usual denotes the parity.

The most important example of Lie superalgebra is the algebra of endomorphisms of a super vector space.

Let $\mathrm{gl}(V)$ be the super vector space of endomorphisms of the super vector space V. If $V = k^{m|n}$, we denote $\mathrm{gl}(V)$ by $\mathrm{gl}(m|n)$. The even part $\mathrm{gl}(m|n)_0$ of $\mathrm{gl}(m|n)$ consists of the matrices with entries in k corresponding to endomorphisms preserving the parity, while the odd part $\mathrm{gl}(m|n)_1$ consists of the matrices corresponding to endomorphisms reversing parity:

$$\mathrm{gl}(m|n) = \mathrm{gl}(m|n)_0 \oplus \mathrm{gl}(m|n)_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

Here A and D are $(m \times m)$ and $(n \times n)$-matrices, and B and C are $(m \times n)$ and $(n \times m)$-matrices, all with entries in k.

Then $\mathrm{gl}(m|n)$ is a Lie superalgebra with bracket

$$[X, Y] = XY - (-1)^{|X||Y|}YX.$$

We now want to define the analogue of the special linear Lie algebra.

Definition A.1.1. We define the *special linear Lie superalgebra*, $\mathrm{sl}(m|n)$ and the *projective special linear Lie superalgebra* $\mathrm{psl}(m|m)$ as

$$\mathrm{sl}(m|n) := \{X \in \mathrm{gl}(m|n) \mid \mathrm{str}(X) = 0\},$$

where $\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}\, A - \mathrm{tr}\, D$, $\mathrm{psl}(m|m) := \mathrm{sl}(m|m)/kI_{2m}$. Here str is the *supertrace* (see also Chapter 1) and I_{2m} is the identity $(2m \times 2m)$-matrix. One can easily check that these are Lie superalgebras with brackets induced by the bracket in $\mathrm{gl}(m|n)$.

As for the even and odd parts, we have

$$\mathrm{sl}(m|n)_0 = \mathrm{sl}_m \oplus \mathrm{sl}_n \oplus k, \qquad \mathrm{sl}(m|n)_1 = V_m \otimes V'_n \otimes k \oplus V'_m \otimes V_n \otimes k,$$
$$\mathrm{psl}(m|m)_0 = \mathrm{sl}_m \oplus \mathrm{sl}_m, \qquad \mathrm{psl}(m|m)_1 = V_m \otimes V'_m \oplus V'_m \otimes V_m,$$

where V_m and V'_m denote respectively the defining representation of sl_m and its dual.

We now seek the analogue for the orthogonal and the symplectic Lie algebras. Classically, given a non-degenerate bilinear form f on a vector space V, we can define the orthogonal and the symplectic Lie algebras in the following way:

$$\mathrm{so}(V) := \{A \in \mathrm{gl}(V) | f(Av, u) = -f(v, Au)\} \text{ if } f \text{ is symmetric,}$$
$$\mathrm{sp}(V) := \{A \in \mathrm{gl}(V) | f(Av, u) = -f(v, Au)\} \text{ if } f \text{ is skew-symmetric.}$$

In the super case, as we shall presently see, the two notions come together, since the condition of symmetry in the odd variables for a bilinear form brings in a minus sign, hence becomes a skew-symmetric condition.

Definition A.1.2. We say that a bilinear form f on a super vector space $V = V_0 \oplus V_1$ is *super symmetric* (or *symmetric* for short) if

$$f(u, v) = (-1)^{|u||v|} f(v, u)$$

for any homogeneous elements $u, v \in V$. We say also that f is *consistent* if $f(u, v) = 0$ for $u \in V_0$ and $v \in V_1$.

Hence, as one can readily see, a super symmetric bilinear form f on V gives a symmetric form f_0 on V_0 and a skew-symmetric form f_1 on V_1.

Definition A.1.3. Let f be a non-degenerate consistent super symmetric bilinear form on V, $\dim V = m|n$. We define the *orthosymplectic Lie superalgebra* as

$$\mathrm{osp}(V) := \{X \in \mathrm{gl}(V) \mid f(Xu, v) = -(-1)^{|X||u|} f(u, Xv)\}.$$

Notice that n has to be even since f defines a non-degenerate skew-symmetric form on V_1.

The standard super symmetric form ϕ in $V = k^{m|n}$ corresponds to the following matrices:

$$\phi = \begin{pmatrix} 0 & I_p & 0 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_q \\ 0 & 0 & 0 & -I_q & 0 \end{pmatrix} \quad \text{if } m = 2p + 1, n = 2q,$$

$$\phi = \begin{pmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix} \quad \text{if } m = 2p, n = 2q.$$

Fixing ϕ as above, the matrix form for the orthosymplectic Lie superalgebra, which we shall denote by $\mathrm{osp}(m|n)$, becomes the following:

1. for $m = 2p + 1, n = 2q$,

$$\mathrm{osp}(m|n) = \left\{ \begin{pmatrix} a & b & u & x & x_1 \\ c & -a^t & v & y & y_1 \\ -v^t & -u^t & 0 & z & z_1 \\ y_1^t & x_1^t & z_1^t & d & e \\ -y^t & -x^t & -z^t & f & -d^t \end{pmatrix} \right\};$$

2. for $m = 2p, n = 2q$,

$$\mathrm{osp}(m|n) = \left\{ \begin{pmatrix} a & b & x & x_1 \\ c & -a^t & y & y_1 \\ y_1^t & x_1^t & d & e \\ -y^t & -x^t & f & -d^t \end{pmatrix} \right\},$$

where $a, b, c, d, e, f, x, x_1, y, y_1, u, v$ are matrices of appropriate size and the matrices b and c are skew-symmetric and e and f are symmetric; all matrices have entries in k.

Notice that if $n = 0$, then $\mathrm{osp}(m|0)$ is the orthogonal Lie algebra B_p or D_p depending on the parity of m, while if $m = 0$, $\mathrm{osp}(0|n)$ is the symplectic Lie algebra C_q.

The decomposition of $\mathrm{osp}(m|n)$ into even and odd parts is

$$\begin{aligned} \mathrm{osp}(m|n)_0 &= B_p \oplus C_q \quad \text{if } m = 2p + 1, n = 2q, \\ \mathrm{osp}(m|n)_0 &= D_p \oplus C_q \quad \text{if } m = 2p, n = 2q, \\ \mathrm{osp}(m|n)_1 &= V_m \otimes V_n, \end{aligned}$$

where again V_m denotes the defining representation of $\mathrm{so}(m)$ and V_n the defining representation of C_q.

Next we have the Lie superalgebras in the *strange series* $P(n)$ and $Q(n)$.

Definition A.1.4. We define the *strange series* $P(n)$ as

$$P(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\} \subset \mathrm{gl}(n+1|n+1),$$

where $A \in \mathrm{sl}(n+1)$, B is symmetric and C skew-symmetric.

The *strange series* $Q(n)$ is defined as follows. Set

$$q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\};$$

$\mathrm{sq}(n)$ are the matrices in $q(n)$ with $\mathrm{tr}(B) = 0$ and $Q(n-1) = \mathrm{psq}(n) = \mathrm{sq}(n)/kI_{2n}$, i.e.,

$$Q(n-1) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid B \in \mathrm{sl}_n \right\} / kI_{2n}.$$

Again one can check that these are well-defined Lie superalgebras with the bracket induced from $\mathrm{gl}(n|n)$.

We have

$$P(n)_0 = \mathrm{sl}(n+1), \quad P(n)_1 = \mathrm{Sym}^2(V_{n+1}) \oplus \bigwedge^2(V'_{n+1}),$$
$$Q(n)_0 = \mathrm{sl}(n+1), \quad Q(n)_1 = \mathrm{ad}(\mathrm{sl}(n+1)),$$

where V_{n+1} and $\mathrm{ad}(\mathrm{sl}(n+1))$ are respectively the defining representation and the adjoint representations of $\mathrm{sl}(n+1)$.

The simple finite-dimensional Lie superalgebras over k fall into several classes. To introduce these classes we need a few more definitions.

Definition A.1.5. Let $\mathfrak{g}$ be a Lie superalgebra (always finite-dimensional). We say that $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and it admits no non-trivial ideals. $\mathfrak{g}$ is *classical* if it is simple and $\mathfrak{g}_1$ is completely reducible as a $\mathfrak{g}_0$-module, where the action is given by the bracket. $\mathfrak{g}$ is *basic* if it is classical and it admits a consistent, non-degenerate, invariant bilinear form, that is to say, there exists a consistent, non-degenerate, bilinear form $\langle\ ,\ \rangle\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle$.

Observation A.1.6. If $\langle\ ,\ \rangle$ is a bilinear invariant form on a simple Lie superalgebra $\mathfrak{g}$, then $\langle\ ,\ \rangle$ is either identically zero or non-degenerate. In fact if $X \in \ker\langle\ ,\ \rangle$, i.e., if X is such that $\langle X, Y\rangle = 0$ for all $Y \in \mathfrak{g}$, we have

$$0 = \langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle.$$

Hence the kernel of $\langle\ ,\ \rangle$ is an ideal, hence it is all of $\mathfrak{g}$ or zero.

The simple Lie superalgebras divide into two main types: the classical type, when the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is completely reducible, and the Cartan type. We make a list of such Lie superalgebras, referring the reader to [44] for all proofs regarding the classification.

1 Classical type. The classical type subdivides further into type 1 and type 2. Type 1 classical superalgebras are those for which $\mathfrak{g}_1$ is not irreducible as $\mathfrak{g}_0$-module and type 2 are those for which $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module.

1.1 Classical type 1. The type 1 superalgebras are

$$A(m|n) := \mathrm{sl}(m+1|n+1), \quad m \neq n,$$
$$A(m|m) := \mathrm{psl}(m+1|m+1),$$
$$C(n) := \mathrm{osp}(2|2n-2), \quad P(n).$$

For these superalgebras $\mathfrak{g}_1$ decomposes into two components as a $\mathfrak{g}_0$-module so that $\mathfrak{g}$ admits a compatible $\mathbb{Z}$-grading (where "compatible" in this case means with respect to the $\mathbb{Z}_2$-grading):

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1, \quad \mathfrak{g}_0 = \mathfrak{g}^0, \quad \mathfrak{g}_1 = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1.$$

Notice that the upper indices refer to the $\mathbb{Z}$-grading, while the lower indices to the $\mathbb{Z}_2$-grading.

For $A(m|n)$ and $C(n)$ we have that $\mathfrak{g}^{-1}$ and $\mathfrak{g}^1$ are dual to each other; in fact,

$$\begin{aligned} A(m|n)_0 &= \mathrm{sl}_{m+1} \oplus \mathrm{sl}_{n+1} \oplus k, \\ A(m|n)^1 &= V_{m+1} \otimes V'_{n+1} \otimes k, \\ A(m|n)^{-1} &= V'_{m+1} \otimes V_{n+1} \otimes k, \\ C(n)_0 = C_{n-1}, \quad C(n)^1 &= V_n, \quad C(n)^{-1} = V'_n \cong V_n, \end{aligned}$$

where V_i always denotes the natural representation of the Lie algebra under consideration and V'_i its dual.

For $\mathfrak{g} = P(n)$, $\mathfrak{g}^{-1}$ and $\mathfrak{g}^1$ are not dual to each other. We have

$$\mathfrak{g}^1 = \mathrm{Sym}^2 V_{n+1}, \quad \mathfrak{g}^{-1} = \bigwedge^2 V'_{n+1}.$$

1.2 Classical type 2. The type 2 superalgebras are those for which $\mathfrak{g}_1$ is irreducible, so that there is no compatible $\mathbb{Z}$-grading. These Lie superalgebras are

$$\begin{aligned} B(m|n) &= \mathrm{osp}(2m+1|2n), \\ D(m|n) &= \mathrm{osp}(2m|2n), \\ D(2,1;\alpha), \quad F(4)&, \quad G(3), \quad Q(n). \end{aligned}$$

$D(2,1;\alpha)$ is a family with a continuous parameter $\alpha \in k \setminus \{0,-1\}$. Two elements $D(2,1;\alpha)$, $D(2,1;\beta)$ of this family are isomorphic if and only if α and β lie in the same orbit under the action of the group of order 6 generated by $\alpha \mapsto -1-\alpha, \alpha \mapsto 1/\alpha$ (see [44], 2.5).

We have $\mathfrak{g}_0 = \mathrm{sl}_2 \oplus \mathrm{sl}_2 \oplus \mathrm{sl}_2$, $\mathfrak{g}_1 = V_2 \otimes V_2 \otimes V_2$. Without going into detail we want to remark that the bracket restricted to $\mathfrak{g}_1 \times \mathfrak{g}_1$ depends on the parameter α, hence different superalgebras $D(2,1;\alpha)$ cannot be recognized by the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module alone.

For the definition of the exceptional Lie superalgebras $F(4)$ and $G(3)$ see [44].

2 Cartan type. Let $\mathrm{Sym}(V)$ denote the symmetric algebra of the super vector space V. If V has dimension $m|n$ and we fix a basis, we can identify $\mathrm{Sym}(V)$ with $A = k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$, the polynomial algebra with m even indeterminates and n odd ones. We define $W(m|n) = \mathrm{Der}(A)$ as the superalgebra of derivations of A. In general $W(m|n)$ is infinite-dimensional, however when $m = 0$, it is finite-dimensional. We write $W(n)$ for $W(0|n)$:

$$W(n) = \Big\{ \sum_{a_{I,j} \in k} a_{I,j} \xi_{i_1} \dots \xi_{i_l} \partial_{\xi_j} \Big\}.$$

$W(n) = W(n)_0 \oplus W(n)_1$, where $W(n)_0$ corresponds to those elements with l odd, while $W(n)_1$ to those with l even. Define $\Theta(n)$ as the associative superalgebra over A generated by $\theta\xi_1, \dots, \theta\xi_n$ with relations $\theta\xi_i \wedge \theta\xi_j = -\theta\xi_j \wedge \theta\xi_i$ $(i \neq j)$. This is a superalgebra with grading induced by $\deg(\theta\xi_i) = 1$. Furthermore, $W(n)$ acts on $\Theta(n)$ by derivations. We define $S(n)$ and $\widetilde{S}(n)$ as the subalgebras of $W(n)$ annihilating certain elements of $\Theta(n)$ called volume forms. Namely,

$$S(n) := \{D \in W(n) \mid D(\theta\xi_1 \wedge \dots \wedge \theta\xi_n) = 0\},$$
$$\widetilde{S}(n) := \{D \in W(n) \mid D((1 + \xi_1\xi_2 \dots \xi_n)\theta\xi_1 \wedge \dots \wedge \theta\xi_n) = 0\} \text{ for even } n.$$

Finally, we define $H(n)$ as the commutator of a subalgebra of $W(n)$ preserving a certain metric, i.e.,

$$H(n) = [\widetilde{H}(n), \widetilde{H}(n)] \quad \text{where } \widetilde{H}(n) := \{D \in W(n) \mid D(d\xi_1^2 + \dots + d\xi_n^2) = 0\}.$$

We end our discussion on simple Lie superalgebras with a theorem:

Theorem A.1.7. *Every simple finite-dimensional Lie superalgebra over k is isomorphic to one of the following:*

(1) *the classical Lie superalgebras, either isomorphic to a simple Lie algebra or to one of the following classical Lie superalgebras:*

$$A(m|n), \quad B(m|n), \quad C(n), \quad D(m|n), \quad P(n), \quad Q(n),$$

for appropriate ranges of m and n,

$$F(4), \quad G(3), \quad D(2,1;\alpha) \ \textit{for } \alpha \in k \setminus \{0, -1\};$$

(2) *the Lie superalgebras of Cartan type:*

$$W(n), \quad S(n), \quad \widetilde{S}(n) \ \textit{for even } n, \quad H(n).$$

In the next section we will discuss the structure of simple Lie superalgebras in more detail with a particular attention to the families A, B, C, and D.

A.2 Root systems

Similarly to the ordinary setting, for Lie superalgebras we have the notion of Cartan subalgebras and the corresponding root decomposition.

Definition A.2.1. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is a nilpotent, self-normalizing Lie subalgebra of $\mathfrak{g}$. If $\alpha \in \mathfrak{h}_0^*$, we define the super vector space

$$\mathfrak{g}_\alpha : = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \text{ for all } h \in \mathfrak{h}_0\}.$$

If $\mathfrak{g}_\alpha \neq \{0\}$ for $\alpha \in \mathfrak{h}_0^* \setminus \{0\}$, we say that α is a *root* and $\mathfrak{g}_\alpha$ is its *root space*. We say that a root α is *even* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\}$, and *odd* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\}$. Notice that with this definition a root can be both even and odd; this can actually happen, as we shall see in Example A.2.2. We denote by Δ the set of all roots.

In the same way as in the ordinary case, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The set of roots $\Delta \subset \mathfrak{h}_0^* \setminus \{0\}$ is the union of even and odd roots: $\Delta = \Delta_0 \cup \Delta_1$, where

$$\Delta_0 = \{\alpha \in \mathfrak{h}_0^* \setminus \{0\} \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\}\}, \quad \Delta_1 = \{\alpha \in \mathfrak{h}_0^* \setminus \{0\} \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\}\}.$$

If $\mathfrak{g}$ is simple, we have $\mathfrak{h} = \mathfrak{h}_0$ with the sole exception of $Q(n)$. Furthermore, it can happen that $\Delta_0 = \Delta_1$, as we shall see in the next example.

Example A.2.2. According to Definition A.1.4 we have

$$q(2) = \left\{ \begin{pmatrix} h_1 & e & \bar{h}_1 & \bar{e} \\ f & h_2 & \bar{f} & \bar{h}_2 \\ \bar{h}_1 & \bar{e} & h_1 & e \\ \bar{f} & \bar{h}_2 & f & h_2 \end{pmatrix} \right\} \subset \mathrm{gl}(2|2).$$

As a super vector space we have that

$$q(2) = \mathrm{span}_k\{h_1, h_2, \bar{h}_1, \bar{h}_2, e, f, \bar{e}, \bar{f}\},$$

where by a slight abuse of notation we denote a matrix with only two non-zero entries by the corresponding letter above. Clearly, $q(2)_0 = \mathrm{span}_k\{h_1, h_2, e, f\}$ and $q(2)_1 = \mathrm{span}_k\{\bar{h}_1, \bar{h}_2, \bar{e}, \bar{f}\}$. The space $\mathfrak{h} = \mathrm{span}_k\{h_1, h_2, \bar{h}_1, \bar{h}_2\}$ is a Cartan subalgebra of $q(2)$. One can readily check that we have two root spaces of dimension $1|1$:

$$\mathfrak{g}_\alpha = \mathrm{span}_k\{e, \bar{e}\}, \quad \mathfrak{g}_{-\alpha} = \mathrm{span}_k\{f, \bar{f}\},$$

where $\alpha(h_1) = 1, \alpha(h_2) = -1$. Therefore,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}, \quad \Delta_0 = \Delta_1.$$

This somewhat awkward behaviour may suggest excluding $Q(n)$ from the treatment, however we believe that the right philosophy is to ask questions for all classical Lie superalgebras. Before we proceed and give some examples of root systems, we want to make some observations on the Cartan–Killing form.

Observation A.2.3. Let $\mathfrak{g}$ be a classical Lie superalgebra. In analogy with the Cartan–Killing form in the ordinary setting, define the bilinear form

$$(x, y) = \mathrm{str}(\mathrm{ad}(x)\,\mathrm{ad}(y)),$$

where $x, y \in \mathfrak{g}$. As one can easily check, this form is symmetric and consistent. However, quite differently from what happens in the classical setting, it is not always non-degenerate, hence zero. In particular its restriction to a Cartan subalgebra of $\mathfrak{g}$ may be degenerate.

For example, on $\mathrm{sl}(m+1|n+1)$, if $\mathfrak{h}$ is the Cartan subalgebra consisting of diagonal matrices, we have

$$(x, y) = 2(m-n)\,\mathrm{str}(xy)$$

for $x = \mathrm{diag}(a_1, \dots, a_{m+n})$ and $y = \mathrm{diag}(a'_1, \dots, a'_{m+n})$ in $\mathfrak{h}$. When $m = n$ the form is identically zero. Furthermore, it factors to a form on the corresponding Cartan subalgebra of $\mathrm{psl}(m|m)$, which is also identically equal to zero.

The fact that the Cartan–Killing form of a classical Lie superalgebra may be degenerate prompts the definition of *basic classical* Lie superalgebras.

Definition A.2.4. A Lie superalgebra $\mathfrak{g}$ is *basic classical* if $\mathfrak{g}$ is simple, $\mathfrak{g}_0$ is reductive, and $\mathfrak{g}$ admits a non-degenerate invariant symmetric consistent bilinear form.

For example, $A(m|n)$, $C(n)$, and $B(m|n)$ are basic simple Lie superalgebras, while $P(n)$ is not. Notice that even though $A(m|m)$ has a degenerate Cartan–Killing form, there exists a non-degenerate invariant symmetric bilinear form on $A(m|m)$.

The following table summarizes the classification of simple Lie superalgebras together with information about the existence of an invariant non-degenerate symmetric bilinear form.

Simple Lie superalgebras		
Classical		Cartan type
Basic	Strange	
$A(m\mid n)$, $B(m\mid n)$, $C(n)$, and $D(m\mid n)$ $D(1,2;\alpha)$, $G(3)$, $F(4)$	$P(n)$, $Q(n)$	$W(n)$, $S(n)$, $\widetilde{S}(n)$, $H(n)$

We now give a brief description of the root systems for the basic classical Lie superalgebras in the families A, B, C, D. As in the ordinary setting every root system Δ admits a *simple system* $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$. The defining property of Π is that every root in Δ is a linear combination of elements of Π with integral all non-positive, or all non-negative, coefficients, and Π is minimal among such subsets of Δ. The elements α_i of Π are called *simple roots*. Unlike the ordinary setting, the elements of Π are not necessarily linearly independent, e.g., when $\mathfrak{g}$ is of type $A(m|m)$. So, once we fix a simple system, we can write $\Delta = \Delta^+ \sqcup \Delta^-$, where Δ^+ is the set of the positive roots, that is the roots $\alpha = m_1\alpha_1 + \dots + m_r\alpha_r$, $m_i \geq 0$, while Δ^+ is the set of the negative roots, that is the roots $\alpha = n_1\alpha_1 + \dots + n_r\alpha_r$, $n_i \leq 0$. From a representation point of view, one may avoid the technicalities related to the superalgebras of type A by replacing them by $\mathrm{gl}(m|m)$. In the remainder of this appendix we will not discuss the fine details distinguishing $\mathrm{gl}(m|m)$ from the rest of the superalgebras of type A. The interested reader will be able to fill in the details or consult [44], [45] for the whole story.

For all of the families A, B, C, D, the Cartan subalgebras are even $\mathfrak{h} = \mathfrak{h}_0$, and we can choose $\mathfrak{h} = \mathfrak{h}_0$ to be the subalgebra of diagonal matrices. For each family we shall give the root system and a choice of a simple system among the various possibilities. Then for such a choice, in the next section we shall build the Cartan matrix and the Dynkin diagram associated with the simple system. For a complete treatment, comprehending all of the simple systems, we refer the reader to [44], 2.5.2, and [33], part 3.

$A(m|n) = \mathrm{sl}(m+1|n+1)$ for $m \neq n$. Let $\epsilon_i, \delta_j \in \mathfrak{h}^*$, $1 \leq i \leq m+1$, $1 \leq j \leq n+1$, defined as $\epsilon_i(\mathrm{diag}(a_1, \dots, a_{m+n+2})) = a_i$, $i = 1, \dots, m+1$, and $\delta_j(\mathrm{diag}(a_1, \dots, a_{m+n+2})) = a_{m+1+j}$, $j = 1, \dots, n+1$.

Root system:

$$\begin{aligned} \Delta &= \{\epsilon_i - \epsilon_j,\ \delta_k - \delta_l,\ \pm(\epsilon_i - \delta_k)\}, \\ \Delta_0 &= \{\epsilon_i - \epsilon_j,\ \delta_k - \delta_l\}, \\ \Delta_1 &= \{\pm(\epsilon_i - \delta_k)\}, \quad 1 \leq i \neq j \leq m+1,\ 1 \leq k \neq l \leq n+1. \end{aligned}$$

Simple root system:

$$\begin{aligned} \Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3, \dots,\ \alpha_{m+1} = \epsilon_{m+1} - \delta_1, \\ \alpha_{m+2} = \delta_1 - \delta_2, \dots,\ \alpha_{m+n+1} = \delta_n - \delta_{n+1}\}. \end{aligned}$$

For $A(n|n)$, $n > 1$, the root system and the simple root system are given by the same formulas as above. It is useful to remember that the elements ϵ_i, δ_j are not linearly independent. If $m \neq n$, there is one relation between them: $\epsilon_1 + \dots + \epsilon_{m+1} = \delta_1 + \dots + \delta_{n+1}$, while for $m = n$ there are two relations: $\epsilon_1 + \dots + \epsilon_{m+1} = \delta_1 + \dots + \delta_{n+1} = 0$.

$B(m|n) = \mathrm{osp}(2m+1|2n)$. The Cartan matrix is the subalgebra of the diagonal matrices:

$$\mathfrak{h} = \{h = \mathrm{diag}(a_1, \dots, a_m, -a_1, \dots, -a_m, 0, b_1, \dots, b_n, -b_1, \dots, -b_n)\}.$$

Define $\epsilon_i, \delta_j \in \mathfrak{h}^*$ as follows: for $h \in \mathfrak{h}$, let $\epsilon_i(h) = a_i$, $i = 1, \dots, m$, and $\delta_j(h) = b_j$, $j = 1, \dots, n$.

Root system, $m \neq 0$:

$$\begin{aligned} \Delta_0 &= \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm\delta_k \pm \delta_l, \pm 2\delta_k\}, \\ \Delta_1 &= \{\pm\epsilon_i \pm \delta_k, \pm\delta_k\}, \quad 1 \leq i \neq j \leq m,\ 1 \leq k \neq l \leq n. \end{aligned}$$

Root system, $m = 0$:

$$\Delta_0 = \{\pm\delta_k \pm \delta_l, \pm 2\delta_k\}, \quad \Delta_1 = \{\pm\delta_k\}, \quad 1 \leq k \neq l \leq n.$$

Simple root system, $m \neq 0$:

$$\Pi = \{\alpha_1 = \delta_1 - \delta_2, \dots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \epsilon_1,$$
$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \dots, \alpha_{m+n-1} = \epsilon_{m-1} - \epsilon_m, \alpha_{m+n} = \epsilon_m\}.$$

Simple root system, $m = 0$:

$$\Pi = \{\alpha_1 = \delta_1 - \delta_2, \dots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n\}.$$

***C*(*n*) = osp(2|2*n* − 2).** The Cartan matrix is again the subalgebra of the diagonal matrices:

$$\mathfrak{h} = \{h = \operatorname{diag}(a_1, -a_1, b_1, \dots, b_{n-1}, -b_1, \dots, -b_{n-1})\}.$$

Define $\epsilon_1, \delta_1, \dots, \delta_{n-1} \in \mathfrak{h}^*$ as follows: for $h \in \mathfrak{h}$, let $\epsilon_1(h) = a_1$, $\delta_1(h) = b_1$, ..., $\delta_{n-1}(h) = b_{n-1}$.

Root system:

$$\Delta_0 = \{\pm 2\delta_k, \pm\delta_k \pm \delta_l\}, \quad \Delta_1 = \{\pm\epsilon_1 \pm \delta_k\}, \quad 1 \leq k \neq l \leq n-1.$$

Simple root system:

$$\Pi = \{\alpha_1 = \epsilon_1 - \delta_1, \alpha_2 = \delta_1 - \delta_2, \dots, \alpha_{n-1} = \delta_{n-2} - \delta_{n-1}, \alpha_n = 2\delta_{n-1}\}.$$

***D*(*m*|*n*) = osp(2*m*|2*n*).** The Cartan matrix is again the subalgebra of the diagonal matrices:

$$\mathfrak{h} = \{h = \operatorname{diag}(a_1, \dots, a_m, -a_1, \dots, -a_m, b_1, \dots, b_n, -b_1, \dots, -b_n)\}.$$

Define $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n \in \mathfrak{h}^*$ as follows: for $h \in \mathfrak{g}$, let $\epsilon_1(h) = a_1$, ..., $\epsilon_m(h) = a_m$, $\delta_1(h) = b_1$, ..., $\delta_n(h) = b_n$.

Root system:

$$\Delta_0 = \{\pm\epsilon_i \pm \epsilon_j, \pm 2\delta_k, \pm\delta_k \pm \delta_l\},$$
$$\Delta_1 = \{\pm\epsilon_i \pm \delta_k\}, \quad 1 \leq i \neq j \leq m, \ 1 \leq k \neq l \leq n.$$

Simple root system:

$$\Pi = \{\alpha_1 = \delta_1 - \delta_2, \dots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \epsilon_1,$$
$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \dots, \alpha_{m+n-1} = \epsilon_{m-1} - \epsilon_m, \alpha_{m+n} = \epsilon_{m-1} + \epsilon_m\}.$$

We end this section with a summary of the properties of root systems for classical Lie superalgebras. For the proofs we refer the reader to [44], Sections 2.5.3–2.5.4.

Proposition A.2.5. *Let* $\mathfrak{g}$ *be a classical Lie superalgebra and let* $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ *be the root decomposition of* $\mathfrak{g}$ *with respect to a Cartan subalgebra* $\mathfrak{h}$. *Then the following hold:*

(1) $\Delta_0 \cap \Delta_1 = \{0\}$ *unless* $\mathfrak{g} = Q(n)$.

(2) $\dim \mathfrak{g}_\alpha = 1$ *unless* $\mathfrak{g} = A(1|1)$, $P(3)$, $Q(n)$.

(3) *On* $\mathfrak{g}$ *there is one and only one, up to a constant factor, non-degenerate invariant symmetric consistent bilinear form* $(\ ,\ )$, *except for* $P(n)$ *and* $Q(n)$ (*where no such form exists*).

(4) *If* $\mathfrak{g} \neq A(1|1), P(n), Q(n)$, *then* $(\ ,\ )$ *determines a non-degenerate pairing of* $\mathfrak{g}_\alpha$ *with* $\mathfrak{g}_{-\alpha}$. *Furthermore, for every* $\alpha \in \Delta$ *there exists an element* $X_\alpha \in \mathfrak{g}_\alpha$ *such that* $[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})h_\alpha$, *where* $h_\alpha \in \mathfrak{h}$ *is determined by* $\alpha(h) = (h_\alpha, h)$, $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$.

(5) $-\Delta_0 = \Delta_0$, *and for* $\mathfrak{g} \neq P(n)$, $\Delta_1 = -\Delta_1$.

(6) *Let* $\alpha \in \Delta$ *and* $c\alpha \in \Delta$. *Then* $c = \pm 1$ *unless* $\alpha \in \Delta_1$ *and* $(\alpha, \alpha) \neq 0$ *when* $c = \pm 1$ *or* $c = \pm 2$.

A.3 Cartan matrices and Dynkin diagrams

Let us now turn to the construction of the Cartan matrix and the Dynkin diagram associated to a classical Lie superalgebra $\mathfrak{g}$ with a simple root system $\Pi = \{\alpha_i\}_{i \in I}$. For each simple root $\alpha_i \in \Pi$, fix elements $e_i \in \mathfrak{g}_\alpha$, $f_i \in \mathfrak{g}_{-\alpha}$ and set $h_i = [e_i, f_i] \in \mathfrak{g}_0$. Clearly h_i is defined up to a constant. If $\alpha_i(h_i) \neq 0$, we fix it by imposing that $\alpha_i(h_i) = 2$.

Definition A.3.1. The *Cartan matrices* A associated to the simple Lie superalgebra $\mathfrak{g}$ and the simple root system Π is defined as

$$A = (a_{ij}) = (\alpha_i(h_j)).$$

We can write the Cartan matrix for all the classical Lie superalgebras whose roots, together with a choice of a simple system, we have described in the previous section.

- $A(m|n)$. In the case of $A(m|n)$ we can choose $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ (where as usual E_{ij} denotes an elementary matrix). One can check immediately that $h_i = E_{ii} - E_{i+1,i+1}$ for $i \neq m+1$, while $h_{m+1} = [e_{m+1}, f_{m+1}] = e_{m+1} f_{m+1} + f_{m+1} e_{m+1} = E_{m+1,m+1} + E_{m+2,m+2}$. The Cartan matrix has the form

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 2 & -1 & 0 & & & \\ 0 & \dots & & -1 & 0 & +1 & \dots & \dots & \dots \\ 0 & \dots & & 0 & -1 & 2 & -1 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 2 \end{pmatrix},$$

where the zero appears in row $(m+1)$ because $\alpha_{m+1}(h_{m+1}) = 0$.

- $B(m|n), m \neq 0$.

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & \dots & 0 & & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 & \dots & 0 & & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 2 & -1 & 0 & & & & \\ 0 & \dots & & -1 & 0 & +1 & \dots & \dots & & \dots \\ 0 & \dots & & 0 & -1 & 2 & -1 & \dots & & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & & & & & -1 & 2 & -1 \\ 0 & \dots & & & & & & 0 & 2 & 2 \end{pmatrix}$$

- $B(0|n)$.

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & \dots & 0 & \dots \\ -1 & 2 & -1 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & & & -1 & 2 & -1 \\ 0 & \dots & & & & 0 & -2 & 2 \end{pmatrix}$$

- $C(n)$.

$$A = \begin{pmatrix} 0 & +1 & 0 & \dots & 0 & \dots & 0 & \dots \\ -1 & 2 & -1 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & & & -1 & 2 & -2 \\ 0 & \dots & & & & 0 & -1 & 2 \end{pmatrix}$$

- $D(m|n)$.

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & \dots & 0 & \dots & & 0 \\ -1 & 2 & -1 & \dots & 0 & \dots & 0 & \dots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & -1 & 0 & +1 & \dots & \dots & \dots & 0 \\ 0 & \dots & & 0 & -1 & 2 & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & & & & & \cdots -1 & 2 & 0 \\ 0 & \dots & & & & & & \cdots -1 & 0 & 2 \end{pmatrix}$$

As it happens in the classical theory, we can associate a Dynkin diagram to any Cartan matrix (except for the case $D(2, 1; \alpha)$[1]) following the rules:

(1) Put as many nodes as simple roots.

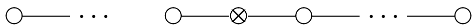
(2) Connect the i-th node with the j-th node with $|a_{ij}a_{ji}|$ links.

(3) The i-th node is *white* if α_i is even, is *black* if α_i is odd and $a_{ii} \neq 0$ and it is *grey* if α_i is odd and $a_{ii} = 0$.

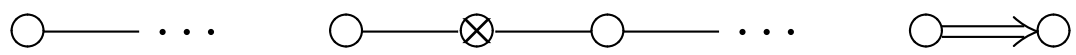
(4) The arrow goes from the long to the short root.

Below we give a list of the Dynkin diagrams for the Lie superalgebras we have described above with respect to the simple system we have written down. It is important to stress that, contrary to the ordinary setting, different Dynkin diagrams may correspond to isomorphic Lie superalgebras. This is ultimately linked to the fact that the Weyl group of $\mathfrak{g}$, defined as the Weyl group of $\mathfrak{g}_0$, does not provide "enough" symmetries for the Lie superalgebra since it is a purely even object.

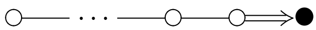
$A(m|n)$



$B(m|n), m > 0$

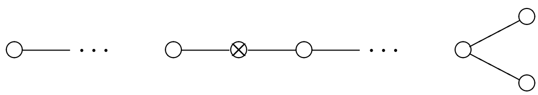


$B(0|n)$



$C(n), n > 2$

$D(m|n)$



We leave it to the reader, as a simple exercise, to verify that these are indeed the Dynkin diagrams associated to the Cartan matrices listed above.

A.4 Classification of finite-dimensional irreducible modules for sl_2, osp(1|2), gl(1|1) and $q(2)$

In the theory of finite-dimensional representations for a classical Lie algebra $\mathfrak{g}$, the representations of sl_2 play a key role. This is due to the fact that, given a root α, the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ corresponding to $\pm\alpha$ generate a subalgebra of $\mathfrak{g}$ isomorphic to sl_2. The conditions for the finite-dimensionality of an irreducible $\mathfrak{g}$-module are essentially derived from the conditions for finite-dimensionality of all the sl_2-modules sitting inside the $\mathfrak{g}$-module. In the super category this is no longer true. Given a root α of $\mathfrak{g}$, the root spaces $\mathfrak{g}_{k\alpha}$ corresponding to all roots proportional to α can generate

[1] As for $D(2, 1; \alpha)$ we invite the reader to consult [44], p. 55 for the convention that has to be used.

different subalgebras of $\mathfrak{g}$. Subalgebras of this type were introduced by Penkov and Serganova in [62]. More precisely, a line ℓ in $\mathfrak{h}_0^*$ is defined as a one-dimensional real subspace of $\mathfrak{h}_0^*$ such that $\ell \cap \Delta \neq \emptyset$. The corresponding line subalgebra $\mathfrak{g}^\ell$ of $\mathfrak{g}$ is defined as the subalgebra of $\mathfrak{g}$ generated by all root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \ell$. Penkov and Serganova proved a theorem characterizing all possible line subalgebras of an arbitrary Lie superalgebra, see [62]. One can check immediately that every line subalgebra of a classical simple Lie superalgebra is isomorphic to one of the following: sl_2, $\mathrm{osp}(1|2)$, $\mathrm{sl}(1|1)$, $q(2)$, or an odd one-dimensional nilpotent superalgebra. The last two occur for $\mathfrak{g}$ of type P or Q. Below is a list of all finite-dimensional irreducible modules of sl_2, $\mathrm{osp}(1|2)$, $\mathrm{gl}(1|1)$, and $q(2)$ (the modules for $\mathrm{sl}(1|1)$ come from a straightforward generalization from those for $\mathrm{gl}(1|1)$).

The representations of the special linear Lie algebra $\mathbf{sl_2}$. The finite-dimensional representations of sl_2 are well known, we just quote the theorem, referring the reader to [75], Ch. IV, for more details. Let $\mathrm{sl}_2 = \mathrm{span}_k\{h, e, f\}$, with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

Theorem A.4.1. *Let V be an irreducible finite-dimensional sl_2-module of dimension $n+1$. Then there exists a basis $v_0, \ldots, v_n$ of V such that*

$$hv_i = (n-2i)v_i, \quad 0 \le i \le n,$$

$$ev_0 = 0, \quad ev_i = (n-i+1)v_{i-1}, \quad 1 \le i \le n,$$

$$fv_n = 0, \quad fv_i = (i+1)v_{i+1}, \quad 0 \le i \le n-1.$$

Conversely, for each positive integer n, there exists exactly one equivalence class of irreducible representations of sl_2 of dimension $n+1$, defined by the action described above.

The vector v_0 is called a *highest weight vector* and spans the one-dimensional subspace of V annihilated by e.

The representations of the orthosymplectic Lie algebra osp(1|2). Let us start with an explicit description of this Lie superalgebra.

$$\mathrm{osp}(1|2) = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ \beta & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ -\alpha & & \end{pmatrix} \mid B \in \mathrm{sl}_2 \right\} \subset \mathrm{gl}(1|2).$$

A basis is given by

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where $\text{span}_k\{e, f, h\} \cong \text{sl}_2$ and the other non-zero brackets are

$$[e, y] = -x, \quad [f, x] = -y, \quad [x, x] = 2e, \quad [y, y] = -2f,$$
$$[x, y] = h, \quad [h, x] = x, \quad [h, y] = -y.$$

The next proposition classifies the irreducible finite-dimensional representations of osp(1|2).

Theorem A.4.2. *Let V be an irreducible finite-dimensional* osp(1|2)*-module. Then V has dimension $2n+1$ and there exists a basis $v_0, \ldots, v_n$, $w_0, \ldots, w_{n-1}$ of V such that*

$$\begin{aligned} hv_i &= (n-2i)v_i, & hw_i &= (n-1-2i)w_i, \\ ev_i &= (n-i+1)v_{i-1}, & ew_i &= (n-i)w_{i-1}, \\ fv_i &= (i+1)v_{i+1}, & fw_i &= (i+1)w_{i+1}, \\ xv_i &= w_{i-1}, & xw_i &= (n-i)v_i, \\ yv_i &= w_i, & xw_i &= -(i+1)v_{i+1}, \end{aligned}$$

where we have assumed that $v_{-1} = w_{-1} = v_{n+1} = w_n = 0$. Up to a change of parity, $V_0 = \text{span}\{v_0, \ldots, v_n\}$ and $V_1 = \text{span}\{w_0, \ldots, w_{n-1}\}$ and they are irreducible sl_2-modules of dimension $n+1$ and n, respectively.

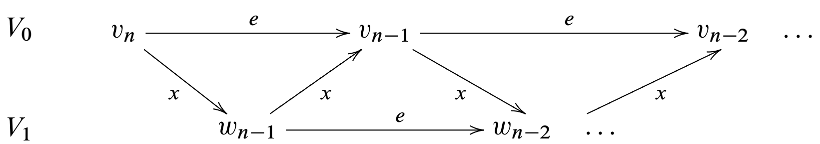
Conversely, for every odd positive integer $2n+1$, up to a change of parity, there exists exactly one equivalence class of irreducible representations of osp(1|2) *of dimension $2n+1$, defined by the action described above.*

Proof. We sketch the proof leaving the details to the reader as an exercise. The parity change in a super vector space commutes with actions. If v_0 is the highest weight vector, it could be $v_0 \in V_0$ or $v_0 \in V_1$, hence without loss of generality we may assume that $v_0 \in V_0$. Applying Theorem A.4.1 we conclude that $ev_0 = 0$ and $hv_0 = nv_0$ for some non-negative integer n. Now set

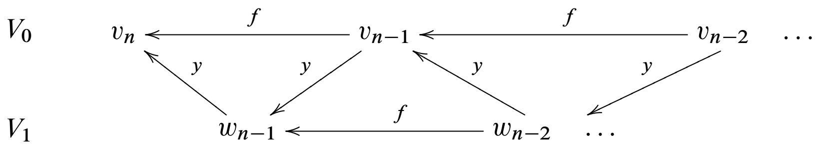
$$v_i = \tfrac{1}{i!} f^i v_0 \quad \text{for } 0 \le i \le n, \quad w_i = \tfrac{1}{i!} f^i y v_0 \quad \text{for } 0 \le i \le n-1.$$

One can now verify that the basis of osp(1|2) acts on the set $v_0, \ldots, v_n, w_0, \ldots, w_{n-1}$ as claimed and that this action provides $V = \text{span}\,\{v_0, \ldots, v_n, w_0, \ldots, w_{n-1}\}$ with a structure of an irreducible osp(1|2)-module.

Schematically we have the following picture:



and similarly



□

The representations of the general linear supergroup gl(1|1). As a super vector space we have

$$\mathrm{gl}(1|1) = \mathrm{span}\left\{h_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ h_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\ x = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ y = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}$$

with brackets

$$[h_1, x] = x, \quad [h_2, x] = -x, \quad [h_1, y] = -y, \quad [h_2, y] = y,$$
$$[x, y] = h_1 + h_2, \quad [x, x] = [y, y] = 0.$$

Theorem A.4.3. *Let V be the irreducible representation of* gl(1|1) *with highest weight* (λ_1, λ_2). *Then:*

(1) *If $\lambda_1 + \lambda_2 \neq 0$, V is $1|1$-dimensional and spanned by v, the highest weight vector, and yv.*

(2) *If $\lambda_1 + \lambda_2 = 0$, V is one-dimensional.*

Proof. Let v be the highest weight vector of V. We have

$$x(yv) = -yxv + [x, y]v = (h_1 + h_2)v = (\lambda_1 + \lambda_2)v$$

and

$$y^2 v = (1/2)[y, y]v = 0, \quad x^2 v = (1/2)[x, x]v = 0,$$

which proves the theorem. For $\lambda_1 + \lambda_2 \neq 0$ the statement is illustrated by the diagram

$$yv \underset{y}{\overset{x}{\rightleftarrows}} v\ .$$

□

The representations of $q(2)$. The irreducible finite-dimensional modules over $\mathfrak{g} = q(2)$ have a more complicated structure. Recall that $\Delta = \{\pm\alpha\}$, and each of $\mathfrak{g}_{\{\pm\alpha\}}$ is $1|1$-dimensional. Furthermore, $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_1$ is $2|2$-dimensional. If we choose the usual basis $\{h_1, h_2\}$ of $\mathfrak{h}_0$, we conclude that every finite-dimensional irreducible $\mathfrak{g}$-module V has a highest weight (λ_1, λ_2). The highest weight space of $V_{(\lambda_1, \lambda_2)}$ is an irreducible $\mathfrak{h}$-module, and it is no longer necessarily one-dimensional. We leave it to the reader to verify that for generic highest weight (λ_1, λ_2), the highest weight space $V_{(\lambda_1,\lambda_2)}$ is an irreducible module of a (suitably chosen) Clifford algebra. In general it is of dimension $1|1$. Since $\mathfrak{g}$ contains a subalgebra isomorphic to gl_2, a necessary condition for finite-dimensionality of V is $\lambda_1 - \lambda_2$ to be a non-negative integer. Surprisingly, it is not enough. The following theorem is proved in [63]:

Theorem A.4.4. *Let $\lambda = (\lambda_1, \lambda_2)$ be a weight of $q(2)$ such that $\lambda_1 - \lambda_2$ is a positive integer or $\lambda_1 = \lambda_2 = 0$. There exists a unique (up to a change of parity) finite-dimensional irreducible $q(2)$-module $V(\lambda)$ with highest weight λ. The weights of $V(\lambda)$ are $(\lambda_1, \lambda_2), (\lambda_1 - 1, \lambda_2 + 1), \dots, (\lambda_2, \lambda_1)$.*

Conversely, every irreducible finite-dimensional $q(2)$-module V is isomorphic to $V(\lambda)$ for some λ as above.

For the proof as well as further study of $q(n)$-modules we refer the reader to the work of Penkov, [63].

A.5 Representations of basic Lie superalgebras

Before starting our description of the finite-dimensional representations of basic Lie superalgebras, we quickly review the classical theory of representations of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a fixed Cartan subalgebra, $\Delta = \Delta^+ \cup \Delta^-$ the root system and V a $\mathfrak{g}$-module. If $\mu \in \mathfrak{h}^*$ we define *weight space* of weight μ as the non-zero vector space:

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

The elements in V_μ are called weight vectors.

The following theorem is crucial in understanding the finite-dimensional representations of a complex simple Lie algebra.

Theorem A.5.1. *Let $\mathfrak{g}$ be a simple Lie algebra and let V be an irreducible finite-dimensional representation. Then we have the following.*

(1) *There exists a non-zero vector v^+ such that it is a weight vector for some weight λ and $X_\alpha v^+ = 0$ for all X_α, $\alpha > 0$, where X_α is a root vector for the root α. The weight λ is called* ***highest weight*** *and v^+* ***highest weight vector****.*

(2) *We have $\dim V_\lambda = 1$, and every weight has the form $\mu = \lambda - \sum_{\alpha>0} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}_{\geq 0}$ for every α. Moreover $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$. (Every representation with this property is called a* ***weight representation****.)*

(3) *The weight λ and the span of v^+ are uniquely determined.*

(4) *We have $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$, where h_α is as in Proposition A.2.5; such a weight λ is called a* ***dominant integral****.*

(5) *Given any dominant integral weight $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible finite-dimensional module V in which λ is the highest weight.*

Proof. See [71], p. 14. □

We now outline an approach to the representation theory of Lie algebras via induced modules. This approach is alternative to the approach with generators and relations and provides an elegant way to prove the existence of a finite-dimensional representation associated to a dominant integral weight.

Let $\lambda \in \mathfrak{h}^*$ and define $\mathfrak{b}^+ := \mathfrak{n}^+ \oplus \mathfrak{h}$ to be the *Borel subalgebra* associated to $\mathfrak{n}^+$ and $\mathfrak{h}$, where $\mathfrak{n}^+$ is the Lie subalgebra generated by the positive roots, once we fix a simple system. The weight λ defines a one-dimensional representation k of $\mathfrak{b}^+$ on which $\mathfrak{h}$ acts via λ and $\mathfrak{n}^+$ acts trivially:

$$(h+n)\cdot v = \lambda(h)v, \quad v = 1 \in k, \quad \text{for all } h \in \mathfrak{h} \text{ and } n \in \mathfrak{n}^+.$$

Definition A.5.2. We define the *induced module*

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{g}} \lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} k.$$

$M(\lambda)$ is also called the *Verma module* associated to the weight λ.

Observation A.5.3. The induced module $M(\lambda) \cong U(\mathfrak{n}^-)$ as $\mathfrak{n}^-$-modules. The isomorphism is given by

$$U(\mathfrak{n}^-) \to M(\lambda), \quad u \mapsto u \otimes 1.$$

This follows immediately from the PBW theorem.

Proposition A.5.4. *Let $\lambda \in \mathfrak{h}^*$, $\{\alpha_1, \dots, \alpha_n\}$ a simple system. Then we have:*

(1) *$M(\lambda) = \bigoplus M(\lambda)_\mu$, i.e., $M(\lambda)$ is a weight representation,*

$$M(\lambda)_\mu = \sum_{p_1,\dots,p_n \in \mathbb{Z}_{\geq 0},\, p_1\alpha_1+\dots+p_n\alpha_n = \lambda-\mu} X_{-\alpha_1}^{p_1} \dots X_{-\alpha_n}^{p_n} \otimes k.$$

(2) *$M(\lambda)_\lambda = 1 \otimes k$, $M(\lambda) = U(\mathfrak{n}^-)M(\lambda)_\lambda$, $U(\mathfrak{n}^+)M(\lambda)_\lambda = 0$.*

Proof. See [27], Ch. 7, Section 1. □

Observation A.5.5. If V is an irreducible finite-dimensional representation of a semisimple $\mathfrak{g}$, there exist a unique $\lambda \in \mathfrak{h}^*$ and a unique morphism $\phi\colon M(\lambda) \to V$ with $\phi(1 \otimes 1) = v$. Hence every irreducible representation V of $\mathfrak{g}$ occurs uniquely as a quotient of a Verma module.

Proposition A.5.6. (1) *Any submodule $F \subset M(\lambda)$ is a weight submodule, i.e., $F = \bigoplus F_\mu$, $F_\mu = F \cap M(\lambda)_\mu$.*

(2) *Every proper submodule of $M(\lambda)$ is contained in*

$$M(\lambda)_+ = \sum_{\mu \neq \lambda} M(\lambda)_\mu.$$

(3) *There exists $K \subset M(\lambda)$ largest proper submodule such that $M(\lambda)/K$ is irreducible (when finite-dimensional $M(\lambda)/K$ is isomorphic to the $V(\lambda)$ in Theorem A.5.1).*

Proof (Sketch). (1) is a direct check. (2) comes from the fact that if $1 \otimes 1 \in F$, then $F = M(\lambda)$. For (3) take K as the sum of all proper submodules. For the complete proof see [27], Ch. 7. □

We have described an alternative way to construct the representations $V(\lambda)$. This is the path taken by Kac in the construction of the irreducible representations of classical Lie superalgebras and as we shall see the construction is very much the same as the ordinary one.

Let $\mathfrak{g}$ be a classical Lie superalgebra. Fix a Cartan subalgebra and a simple system and let $\mathfrak{n}^+$ be as above. Let $\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$ be a fixed Borel subalgebra and $\lambda \in \mathfrak{h}^*$. As before we have a one-dimensional representation of $\mathfrak{b}^+$:

$$(h + n) \cdot v = \lambda(h)v, \quad v = 1 \in k, \quad h \in \mathfrak{h},\ n \in \mathfrak{n}^+.$$

We define

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{g}} \lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} k.$$

Reasoning as in Proposition A.5.6, we have that $M(\lambda)$ contains a unique maximal submodule K such that $M(\lambda)/K = V(\lambda)$ is irreducible. Since change of parity is not an isomorphism of $\mathfrak{g}$-modules, we will always assume that the highest weight space is even, and all statements of uniqueness below are up to a change of parity.

Proposition A.5.7. *Let the notation be as above.*

(1) *The vector v is the unique vector in $M(\lambda)$ (up to a constant) such that $\mathfrak{n}^+ v = 0$.*
(2) *$V(\lambda_1) \cong V(\lambda_2)$ if and only if $\lambda_1 = \lambda_2$.*
(3) *If V is a quotient of $M(\lambda)$, then V has a weight decomposition*

$$V = \bigoplus V_\mu.$$

(4) *Any finite-dimensional irreducible representation of $\mathfrak{g}$ is isomorphic to $V(\lambda)$, for some λ.*

The proof of this proposition is similar to the one in the ordinary setting and more details can be found in [45].

We are ready for the main result on the representation of classical Lie superalgebras.

Let $\mathfrak{g}$ be a basic Lie superalgebra. Fix a Cartan subalgebra and a simple system $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Let $h_i = h_{\alpha_i}$ as in Proposition A.2.5, i.e., $(h_i, h) = \alpha(h)$.

Definition A.5.8. If λ is in $\mathfrak{h}^*$ we define the *numerical marks* $a_i = \lambda(h_i)$.

Theorem A.5.9. *Let the notation be as above. In particular we fix a simple root system for each Lie superalgebra as in Section A.2 or in [44]. Then $V(\lambda)$ is a finite-dimensional representation if and only if:*

(1) *$a_i \in \mathbb{Z}_{\geq 0}$, $i \neq s$, where $s = m + 1$ for $A(m|n)$, $s = m$ for $B(m|n)$, $D(m|n)$ and $s = 1$ for $C(n)$, $F(4)$, $G(3)$, $D(2, 1; \alpha)$.*

(2) *$k \in \mathbb{Z}_{\geq 0}$ where k is given by*

- *For $B(0|n)$, $k = (1/2)a_n$.*
- *For $B(m|n)$, $m > 0$, $k = a_n - a_{n+1} - \cdots - a_{m+n-1} - \frac{1}{2}a_{m+n}$.*
- *For $D(m|n)$, $k = a_n - a_{n+1} - \cdots - a_{m+n-2} - \frac{1}{2}(a_{m+n-1} + a_{m+n})$.*
- *For $D(2,1;\alpha)$, $k = (1+\alpha)^{-1}(2a_1 - a_2 - \alpha a_3)$.*
- *For $F(4)$, $k = (1/3)(2a_1 - 3a_2 - 4a_3 - 2a_4)$.*
- *For $G(3)$, $k = (1/2)(a_1 - 2a_2 - 3a_3)$.*

(3) *Let b be an integer as follows:*

	$B(0\|n)$	$B(m\|n)$	$D(m\|n)$	$D(2,1;\alpha)$	$F(4)$	$G(3)$
b	0	m	m	2	4	3

There are the following supplementary conditions if $k < b$:

- *For $B(m|n)$, $a_{k+n+1} = \cdots = a_{m+n} = 0$.*
- *For $D(m|n)$, if $k < m-1$, $a_{k+n+1} = \cdots = a_{m+n} = 0$; if $k = m-1$, $a_{k+n+1} = a_{m+n}$.*
- *For $D(2,1;\alpha)$, if $k = 0$, $a_i = 0$ for all i; if $k = 1$, $(a_3+1)\alpha = \pm(a_2+1)$.*
- *For $F(4)$, if $k = 0$, $k \neq 1$, $a_i = 0$ for all i; if $k = 2$ $a_2 = a_4 = 0$; if $k = 3$, $a_2 = 2a_4 + 1$.*
- *For $G(3)$, if $k = 0$, $k \neq 1$ $a_i = 0$; if $k = 2$, $a_2 = 0$.*

Conditions (1) and (2) are very natural – these are simply the conditions for λ to be a dominant integral weight of $\mathfrak{g}_0$. Condition (3) is more interesting. One way of understanding it is the following: $V(\lambda)$ is a finite-dimensional module if and only if it is a highest weight module with respect to every Borel subalgebra of $\mathfrak{g}$. In the classical setting all Borel subalgebras of a Lie algebra are conjugate via the elements of the Weyl group W and, if $\mathfrak{b}' = w\mathfrak{b}$ for $w \in W$, then $V_{\mathfrak{b}}(\lambda) = V_{w\mathfrak{b}}(w\lambda)$. Furthermore, the weight λ is dominant and integral with respect to $w\mathfrak{b}$ if and only if λ is dominant integral with respect to $\mathfrak{b}$. For Lie superalgebras it is no longer true that all Borel subalgebras are conjugate. Penkov and Serganova introduced in [62] the so-called "odd reflections" which transform Borel subalgebras and act on weights; these take the place of the missing symmetries of the Weyl group, which is an even object and for this reason does not contain enough reflections to act transitively on the set of Borel subalgebras. The action on weights, however, is no longer a group action. In the remainder of this section we present the main idea and illustrate it on a relevant example.

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and let α be a simple root of $\mathfrak{b}$. Consider the line $\ell = \mathbb{R}\alpha \cap \Delta$, (see the definition of a line at the beginning of Section A.4). The Borel algebra $\mathfrak{b}$ determines a partition $\Delta = \Delta^+ \sqcup \Delta^-$. Set $\ell^{\pm} = \ell \cap \Delta^{\pm}$ and let $(\Delta^{\pm})' = (\Delta^{\pm} \backslash \ell^{\pm}) \cup \ell^{\mp}$. Let $\mathfrak{b}'$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to the partition $\Delta = (\Delta^+)' \sqcup (\Delta^-)'$. If α is odd and the subalgebra generated by X_α and $X_{-\alpha}$ is isomorphic to $\mathrm{sl}(1|1)$, the map $\mathfrak{b} \mapsto \mathfrak{b}'$ is the odd reflection along ℓ that we denote by r_ℓ. Notice that, if α is an even root, then $\mathfrak{b}' = s_\alpha \mathfrak{b}$, $s_\alpha \in W$, otherwise $\mathfrak{b}' = r_\ell \mathfrak{b}$. If $V_{\mathfrak{b}}(\lambda)$ is finite-dimensional, then $V_{\mathfrak{b}}(\lambda) = V_{\mathfrak{b}'}(\lambda')$ for some weight λ'. (If α is even, then $\lambda' = s_\alpha \lambda$.) We write $\lambda' = r_\ell \lambda$. The weight λ' is easily computable - it is simply the lowest weight of the $\mathfrak{g}^\ell + \mathfrak{h}$-module with highest weight λ, where $\mathfrak{g}^\ell$ is the line subalgebra corresponding to the line ℓ (see Section A.4). In the previous section we described all irreducible modules over the possible line subalgebras $\mathfrak{g}^\ell$. We can now rephrase these results as follows.

Theorem A.5.10. *Let $\ell = \mathbb{R}\alpha$ be a simple line of $\mathfrak{b} \subset \mathfrak{g}$, where $\mathfrak{g}$ is a classical simple Lie superalgebra. Assume that $\lambda \in \mathfrak{h}_0^*$ is a weight such that the irreducible $\mathfrak{g}^\ell + \mathfrak{h}$-module with highest weight λ is finite-dimensional. Then $r_\ell \lambda = s_\alpha \lambda$ if $\mathfrak{g}^\ell$ is isomorphic to* sl_2, $\mathrm{osp}(1|2)$, *or* $\mathrm{sq}(2)$. *If $\mathfrak{g}^\ell$ is one-dimensional nilpotent, then $r_\ell \lambda = \lambda$. If $\mathfrak{g}^\ell$ is isomorphic to* $\mathrm{sl}(1|1)$, *then*

$$r_\ell \lambda = \begin{cases} \lambda - \alpha & \textit{if } \lambda(h_\alpha) \neq 0, \\ \lambda & \textit{if } \lambda(h_\alpha) = 0. \end{cases}$$

We have now a stronger necessary condition for the finite-dimensionality of the module $V_{\mathfrak{b}}(\lambda)$. Namely, the weight $r_{\ell_1} r_{\ell_2} \dots r_{\ell_k} \lambda$ has to be dominant with respect to the Borel subalgebra $r_{\ell_1} r_{\ell_2} \dots r_{\ell_k} \mathfrak{b}$ for any sequence of reflections $r_{\ell_1}, r_{\ell_2}, \dots, r_{\ell_k}$ applied subsequently to $\mathfrak{b}$. This condition is also sufficient. We refer the reader to [62] for a detailed proof. This statement provides an effective but not very efficient method for determining the weights λ for which $V_{\mathfrak{b}}(\lambda)$ is finite-dimensional. Serganova in [67] describes a way of finding the minimal number of verifications necessary in the case when $\mathfrak{g}$ is a Kac–Moody superalgebra. We will not give the detailed statement here. Instead, we will use Theorem A.5.10 above to explain condition (3) of Theorem A.5.9 in the case when $\mathfrak{g}$ is of type $B(m|n)$.

Example A.5.11. We want to show that the conditions (1), (2), (3) in Theorem A.5.9 are necessary conditions in order to have finite-dimensional representations for the Lie superalgebra $B(m|n)$. Recall that the root system is (see Section A.2)

$$\Delta_0 = \{\pm\delta_i \pm \delta_j,\ \pm 2\delta_i, \pm\epsilon_i \pm \epsilon_j,\ \pm\epsilon_i\},$$
$$\Delta_1 = \{\pm\delta_i \pm \epsilon_i,\ \pm\delta_i\}.$$

Let us choose a Borel subalgebra, i.e., a simple system

$$\alpha_1 = \delta_1 - \delta_2,\ \alpha_2 = \delta_2 - \delta_3, \dots, \alpha_{n-1} = \delta_{n-1} - \delta_n,$$
$$\alpha_n = \delta_n - \epsilon_1,\ \alpha_{n+1} = \epsilon_1 - \epsilon_2, \dots, \alpha_{n+m} = \epsilon_m.$$

Assume that we have a finite-dimensional representation of highest weight Λ:

$$\Lambda = \mu_1\delta_1 + \dots + \mu_n\delta_n + \lambda_1\epsilon_1 + \dots + \lambda_m\epsilon_m.$$

Let $a_i = \Lambda(h_i)$ be the numerical marks. We have

$$\begin{aligned} a_i &= \mu_i - \mu_{i+1}, \quad 1 \leq i \leq n-1, \\ a_n &= \mu_n + \lambda_1, \\ a_{n+j} &= \lambda_j - \lambda_{j+1}, \quad 1 \leq j \leq m-1, \\ a_{m+n} &= 2\lambda_n. \end{aligned}$$

Since $V(\lambda)$ is a finite-dimensional $B(m|n)_0$ representation, we have

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0,$$

and these are integers or half integers. Hence

$$a_1 \geq 0,\, a_2 \geq 0, \ldots, a_{n-1} \geq 0, \quad a_{n+1} \geq 0,\, a_{n+2} \geq 0, \ldots, a_{n+m} \geq 0$$

are non-negative and integral. This gives the condition (1) in Theorem A.5.9. Now observe that

$$\mu_n = a_n - a_{n+1} - a_{n+2} - \cdots - a_{n+m-1} - (1/2)a_{n+m}.$$

The fact $\mu_n \in \mathbb{Z}_{\geq 0}$ gives condition (2). Now we want to see how to get condition (3).

In order to clarify the construction we work in $B(2|1)$, but the reader can immediately see how easily this can be generalized to $B(m|n)$.

In this case our weight Λ and our fixed Borel subalgebra $\mathfrak{b}$ (i.e., simple system) are

$$\Lambda = \mu_1\delta_1 + \lambda_1\epsilon_1 + \lambda_2\epsilon_2, \quad \{\delta_1 - \epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_2\}.$$

In the ordinary setting we have that the Weyl group acts transitively on the set of Borel subalgebras, hence the weight $w \cdot \Lambda$ is automatically dominant and integral with respect to the Borel subalgebra $w \cdot \mathfrak{b}$. Here however, we have fewer symmetries since the Weyl group is not a super object, but remains ordinary. In particular not all Borel subalgebras are conjugate with respect to the Weyl group. We can compensate the lack of enough symmetries of the Weyl group, by using odd reflections. In this way we can reach every Borel subalgebra starting from a fixed one. For this reason we are going to use an action of gl(1|1), whose representations we have studied in Section A.4.

Our goal is to use odd reflections to move the chosen Borel subalgebra to the one with the associated simple system: $\{\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_1, \delta_1\}$, while tracking down how the weight Λ transforms. The dominant integral condition on the new weight will give us the additional conditions (3).

Let us start with an odd reflection along the root $\delta_1 - \epsilon_1$. This corresponds to a gl(1|1) representation of highest weight $\lambda_1 + \mu_1$. By Theorem A.4.3 we have only two cases:

(1) $\lambda_1 + \mu_1 = 0$, i.e., $\lambda_1 = \mu_1 = 0$ (since they are both positive), corresponding to a 1-dimensional representation,

(2) $\lambda_1 + \mu_1 \neq 0$ corresponding to a $1|1$-dimensional representation.

In case $\lambda_1 + \mu_1 = 0$, we have that Λ does not change, while in case $\lambda_1 + \mu_1 \neq 0$, Λ becomes $\Lambda' = \Lambda - (\delta_1 - \epsilon_1) = (\mu_1 - 1)\delta_1 + (\lambda_1 + 1)\epsilon_1 + \lambda_2\epsilon_2$.

We now apply another odd reflection: along $\delta_1 - \epsilon_2$. Again we are faced with two possibilities as before. We are going to see what happens separately in the two cases discussed above.

- Case $\lambda_1 = \mu_1 = 0$, $\Lambda = \mu_1\delta_1 + \lambda_1\epsilon_1 + \lambda_2\epsilon_2$.
  i) If $\lambda_2 = \mu_1 = 0$, then $\Lambda = 0$, which is the trivial case.
  ii) If $\lambda_2+\mu_1 \neq 0$, we reach a contradiction since $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_1 = \mu_1 = 0$.

- Case $\lambda_1 + \mu_1 \neq 0$, $\Lambda' = (\mu_1 - 1)\delta_1 + (\lambda_1 + 1)\epsilon_1 + \lambda_2\epsilon_2$.
  iii) If $\lambda_2 + \mu_1 - 1 = 0$, the weight Λ' is unchanged and since $\mu_1 - 1 \geq 0$, $\lambda_2 + \mu_1 - 1 \geq 0$, we have $\mu_1 = 1$, $\lambda_2 = 0$.
  iv) If $\lambda_2 + \mu_1 - 1 \neq 0$ from Λ' we obtain a new weight $\Lambda'' = (\mu_1 - 2)\delta_1 + (\lambda_1 + 1)\epsilon_1 + (\lambda_2 + 1)\epsilon_2$. Hence we obtain the condition $\mu_1 \geq 2$.

Let us now take a closer look at condition (3) in Theorem A.5.9.

We have $k = a_1 - a_2 - \frac{1}{2}a_3 = \mu_1$ and $b = m = 2$. If $k = \mu_1 \geq 2$ then there are no extra conditions and we are in the last case.

If $k = \mu_1 < 2$, then we are in one of the three previous cases, i.e., at some point at least once we have a trivial gl$(1|1)$ representation occurring. If $k = \mu_1 = 0$ then $\lambda_1 = \lambda_2 = \mu_1 = 0$, i.e., $a_2 = \lambda_1 - \lambda_2 = a_3 = 2\lambda_2 = 0$. If $k = \mu_1 = 1$ we see $a_3 = 2\lambda_2 = 0$.

This proves only of course that the condition (3) is necessary. However, using Theorem 10.5 in [67] we can conclude that it is also sufficient.

A.6 More on representations of Lie superalgebras

The aim of this short section is to mention some properties of superalgebras and their representations which differ from the classical setting. We do not aim to give a complete and broad view of the subject and our references are extremely limited.

The structure of Lie superalgebras beyond the simple finite-dimensional ones exhibits some unexpected properties. The first example of such a difference is the fact that a semisimple Lie superalgebra, i.e., a superalgebra whose radical is trivial, is not necessarily a direct sum of simple ones, see [44]. Another surprising fact, related to Schur's lemma below, is that the tensor product $V_1 \otimes V_2$ of two irreducible modules over $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively is not necessarily irreducible as a module over $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Following the classical theory of Lie algebras, the simple finite-dimensional real Lie superalgebras have been classified, [44].

Not every simple finite-dimensional Lie superalgebra exhibits properties analogous to the properties of Kac–Moody Lie algebras. Even the superalgebras that do have such properties admit several Cartan matrices and corresponding Dynkin diagrams. This feature of superalgebras has made the problem of identifying good classes of infinite-dimensional Lie superalgebras non-trivial. Kac and Wakimoto introduced certain affine Lie superalgebras and wrote conjectural character formulas for their highest weight modules. Then they showed how these character formulas imply denominator identities which generalize Macdonald's identities for affine Lie algebras. Despite the significance of such identities, little progress has been made towards proving the respective character formulas. Serganova in her recent paper [67] proposed a definition of a Kac–Moody superalgebra which addresses the differences between superalgebras and their ordinary counterparts. Apart from being an important foundational paper, [67] is also an excellent review of the current state of the art results on Kac–Moody superalgebras and initiates the study of their representations.

The representation theory of Lie superalgebras is much richer than the one of Lie algebras and, despite the great progress made since the pioneering paper [45], many important questions remain open. Kac noticed, [45], that unlike the ordinary case, the category of finite-dimensional representations of a simple Lie superalgebra is not semisimple. That is, not every finite-dimensional module is a direct sum of irreducible modules. Kac also singled out a class of irreducible representations, the so-called *typical representations*, whose properties closely resemble the properties of irreducible finite-dimensional modules over simple Lie algebras. Typical representations always split in finite-dimensional representations and it is not difficult to determine their characters, etc. The rest of the irreducible representations, the atypical ones, are far more difficult and understanding even their characters is a very challenging problem. This problem has now been solved for the cases of $\mathrm{gl}(m|n)$, $q(n)$, and $\mathrm{osp}(m|2n)$. There are two approaches.

The first one, geometric in nature, is a very sophisticated generalization of its ordinary counter part. It took more than twenty years until a complete solution for the above mentioned superalgebras was obtained. The main contributors towards this approach were Penkov and Serganova who in the late 1980s and 1990s developed the foundations of the theory and proved many particular cases. For $\mathrm{gl}(m|n)$ the character formula was obtained by Serganova, [68], followed by a work of Penkov and Serganova on the characters of $q(n)$. Finally, Gruson and Serganova, [42], proved the character formula for $\mathrm{osp}(m|2n)$. For more detail on this approach we refer the reader to [42].

The second approach uses super analogs of Schur and Howe duality to reduce the problem of determining the characters of irreducible finite-dimensional representations of the Lie superalgebras $\mathrm{gl}(m|n)$, $q(n)$, and $\mathrm{osp}(m|2n)$ to a related problem about representations of infinite-dimensional Lie algebras. The solution of the latter problem is given in terms of the celebrated Kazhdan–Lusztig polynomials. This approach was first introduced by Brundan for $\mathrm{gl}(m|n)$ and later developed further by, among others, Brundan, Stroppel, Cheng, Lam, Wang, and Zhang. We refer the interested reader to [21], and the references therein, for more detail on this approach.

Any detailed discussion of the intricacies of the representation theory of Lie superalgebras is beyond the scope of this Appendix. However, we would like to illustrate some features on the case of $\mathrm{gl}(m|n)$. In the ordinary setting, the Verma module $M(\lambda)$ for dominant integral λ has a unique finite-dimensional quotient. This is no longer true for $\mathrm{gl}(m|n)$. If λ is an atypical dominant integral weight, i.e., if $V(\lambda)$ is an atypical finite-dimensional module, $M(\lambda)$ has more than one finite-dimensional quotient. In fact, there is a universal finite-dimensional quotient $K(\lambda)$ called the *Kac module* $K(\lambda)$, first introduced by Kac. The relationship between $V(\lambda)$ and $K(\lambda)$ somewhat resembles the relationship between $V(\lambda)$ and $M(\lambda)$ for infinite-dimensional modules $V(\lambda)$ over a simple Lie algebra. Hence the problem of finding the character of $V(\lambda)$ for atypical λ is somewhat close to the problem of finding the characters of the irreducible modules in the category $\mathcal{O}$ of highest weight modules over Lie algebras.

A.7 Schur's lemma

We end our brief treatment of representation theory with the super version of Schur's lemma, which is a fundamental result.

Theorem A.7.1 (Schur's lemma). *Let $\mathcal{M}$ be a subset of $\mathrm{gl}(m|n)$ acting in an irreducible way on a super vector space V. Let $C(\mathcal{M})$ be the subalgebra of $\mathrm{gl}(m|n)$ consisting of the endomorphisms supercommuting with $\mathcal{M}$:*

$$C(\mathcal{M}) := \{a \in \mathrm{gl}(m|n) \mid [a, b] = 0 \textit{ for all } b \in \mathcal{M}\}.$$

Then one and only one of these two possibilities arises:

(1) $C(\mathcal{M}) = \langle 1 \rangle$,

(2) $\dim V_0 = \dim V_1$, $C(\mathcal{M})_0 = \langle 1 \rangle$, $C(\mathcal{M})_1 = \langle A \rangle$, *where $A \neq 0$ permutes V_0 and V_1 and $A^2 = 1$.*

Proof. Let F be an even endomorphism commuting with $\mathcal{M}$. By the classical Schur's lemma, we have $F = cI$, where $c \in k$. Let F be an odd morphism commuting with $\mathcal{M}$. Then F^2 is even, hence we apply the classical Schur lemma to obtain $F^2 = cI$. If G is another such morphism, it follows that FG is even, hence $FG = cI$. Thus $G = F^{-1} = F$ up to a scalar. □

We now want to show that the two possibilities which arise in the super version of Schur's lemma, do really occur even in very simple examples. We want to stress the fact that the hypothesis of irreducibility of the representation means that there are no *graded* $\mathfrak{g}$-submodules of V.

In our example we shall take $\mathcal{M} = \rho(\mathfrak{n})$, where ρ is a representation of the *Heisenberg superalgebra* $\mathfrak{n}$, which is an interesting object in itself, given its importance in physics.

Example A.7.2. Let us define the complex Heisenberg superalgebra $\mathfrak{n}$ with even (central) generator e and odd generators $a_1, \dots, a_n, b_1, \dots, b_n$. The only non-zero brackets are

$$[a_i, b_i] = e.$$

Let $V = \bigwedge(\xi_1, \dots, \xi_n)$ denote the complex exterior algebra with generators $\xi_1, \dots, \xi_n$. This is a super vector space, where the even part is generated by products of an even number of ξ_i's while the odd part, by products of an odd number of the ξ_i's. We define a representation ρ_α of $\mathfrak{n}$ on $\bigwedge(\xi_1, \dots, \xi_n)$ in the following way:

$$a_i \cdot u = \partial_{\xi_i} u, \quad b_i \cdot u = \alpha \xi_i u, \quad e \cdot u = \alpha u,$$

where α is a fixed non-zero complex number. One can easily check that ρ_α is well defined, in other words, the brackets are preserved. We now want to show that ρ_α is irreducible and $C(\rho(\mathfrak{n}))$ the endomorphisms of V supercommuting with the endomorphisms induced by this action are multiples of the identity. We will do this in the case of $n = 2$, leaving the generalization to the reader as an exercise. For the irreducibility, notice that the even and odd parts have dimension 2. Hence a proper invariant subspace must have even or odd dimension equal to 1 and one can check immediately by direct inspection that this is not possible.

Let $\phi \in \mathrm{gl}(V)$, $V = \bigwedge(\xi_1, \xi_2)$. By Schur's lemma, in order to prove $\phi = \lambda I$ it is enough to prove that ϕ preserves parity. By contradiction assume that ϕ does not. Again by Schur's lemma,

$$\phi = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where B and C are two by two invertible matrices and we have chosen the (graded) basis $1, \xi_1 \wedge \xi_2, \xi_1, \xi_2$. By hypothesis we have

$$\partial_{\xi_1}(\phi(1)) = -\phi(\partial_{\xi_1}(1)) = 0, \quad \partial_{\xi_1}(b_{11}\xi_1 + b_{21}\xi_2) = b_{11} = 0.$$

Similarly by commuting ϕ with ∂_{ξ_2} one can see that $b_{21} = 0$. This gives us a contradiction since by Schur's lemma, ϕ must be invertible.

Let us now turn to another example, which realizes the second possibility of Schur's lemma. Consider $\mathfrak{n}' = \mathfrak{n} \oplus \langle c \rangle$ where the odd element c satisfies

$$[\mathfrak{n}, c] = 0, \quad [c, c] = e.$$

Consider the vector space $V = \bigwedge(\xi_1, \dots, \xi_n) \otimes k(\epsilon)$, where $k(\epsilon) = T(k^{0|1})/(\epsilon \otimes \epsilon - \alpha/2)$, and $T(k^{0|1})$ denotes the full tensor algebra over k of the super vector space $k^{0|1}$ with (canonical) basis ϵ. As a super vector space $k(\epsilon) = \{a + b\epsilon\} \cong k^{1|1}$. Define on V the action ρ'_α:

$$\rho'_\alpha(h)(u \otimes v) = \rho_\alpha(h)u \otimes v \quad \text{for all } h \in \mathfrak{n}, \quad \rho'_\alpha(c)(u \otimes v) = (1 \otimes \epsilon)(u \otimes v).$$

We leave it to the reader to check that this is a well-defined action and that it is irreducible. If we take $n = 2$ we can directly check that the odd morphism

$$\phi = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}$$

commutes with the action ρ'_α for $\alpha = 2$, thus realizing (2) in Schur's lemma.

B

Categories

In this appendix we collect some basic facts on categories, sheafification of functors and commutative algebra that are needed in our notes. All of this material is well known; we have made an effort to make the appendix self-contained and to provide precise references for those results that we are unable to prove here.

B.1 Categories

We want to make a brief summary of formal properties and definitions relative to categories. For more details one can see for example [51].

Definition B.1.1. A *category* $\mathcal{C}$ consists of a collection of objects, $\mathrm{Ob}(\mathcal{C})$, and sets of *morphisms* between objects. For all pairs $A, B \in \mathrm{Ob}(\mathcal{C})$, we denote the set of morphisms from A to B by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ so that, for all $A, B, C \in \mathcal{C}$, there exists an association

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

called the "composition law" ($(f, g) \to f \circ g$), which satisfies the properties:

(i) the law "$\circ$" is associative,

(ii) for all $A, B \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ so that $f \circ \mathrm{id}_A = f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{id}_A \circ g = g$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$,

(iii) $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A', B')$ are disjoint unless $A = A'$, $B = B'$, in which case they are equal.

If a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is invertible; in other words, there exists another morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $f \circ g$ and $g \circ f$ are the identities, respectively, in $\mathrm{Hom}_{\mathcal{C}}(B, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A, A)$, we say that f is an *isomorphism*.

Once the category is understood, it is conventional to write $A \in \mathcal{C}$ instead of $A \in \mathrm{Ob}(\mathcal{C})$ for objects. We may also suppress the "$\mathcal{C}$" from $\mathrm{Hom}_{\mathcal{C}}$ and just write Hom whenever there is no danger of confusion.

Essentially a category is a collection of objects which share some basic structure, along with maps between objects which preserve that structure.

Examples B.1.2. (1) Let (sets) denote the category of sets. The objects are the sets, and for any two sets $A, B \in \mathrm{Ob}((\text{sets}))$, the morphisms are the maps from A to B.

(2) Let $\mathcal{G}$ denote the category of groups. Any object $G \in \mathcal{G}$ is a group, and for any two groups $G, H \in \mathrm{Ob}(\mathcal{G})$, the set $\mathrm{Hom}_{\mathcal{G}}(G, H)$ is the set of group homomorphisms from G to H.

Definition B.1.3. A category $\mathcal{C}'$ is a *subcategory* of category $\mathcal{C}$ if $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{C}'$, so that the composition law "$\circ$" on $\mathcal{C}'$ is induced by that on $\mathcal{C}$.

Example B.1.4. The category $\mathcal{A}$ of *abelian groups* and group morphisms is a subcategory of the category of groups $\mathcal{G}$.

Definition B.1.5. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. Then a *covariant* resp. *contravariant functor* $F : \mathcal{C}_1 \to \mathcal{C}_2$ consists of:

(1) a map $F\colon \mathrm{Ob}(\mathcal{C}_1) \to \mathrm{Ob}(\mathcal{C}_2)$ and

(2) a map (denoted by the same F) $F\colon \mathrm{Hom}_{\mathcal{C}_1}(A, B) \to \mathrm{Hom}_{\mathcal{C}_2}(F(A), F(B))$ resp. $F\colon \mathrm{Hom}_{\mathcal{C}_1}(A, B) \to \mathrm{Hom}_{\mathcal{C}_2}(F(B), F(A))$ so that

   (i) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and

   (ii) $F(f \circ g) = F(f) \circ F(g)$ resp. $F(f \circ g) = F(g) \circ F(f)$ for all $A, B \in \mathrm{Ob}(\mathcal{C}_1)$.

By "functor" we always mean covariant functor. A contravariant functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is the same as a covariant functor from $\mathcal{C}_1^{\mathrm{op}} \to \mathcal{C}_2$, where $\mathcal{C}_1^{\mathrm{op}}$ denotes the *opposite* category, i.e., the category where all morphism arrows are reversed.

Definition B.1.6. Let F_1, F_2 be two functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. We say that there is a *natural transformation* of functors $\varphi\colon F_1 \to F_2$ if for all $A \in \mathcal{C}_1$ there is a set of morphisms $\varphi_A\colon F_1(A) \to F_2(A)$ so that for any $f \in \mathrm{Hom}_{\mathcal{C}_1}(A, B)$ $(B \in \mathcal{C}_1)$, the following diagram commutes:

$$\begin{array}{ccc} F_1(A) & \xrightarrow{\varphi_A} & F_2(A) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(B) & \xrightarrow{\varphi_B} & F_2(B). \end{array} \tag{B.1}$$

The family of functions φ_A is called *functorial* in A.

We say that two functors $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ are *isomorphic* if there exist two natural transformations $\phi\colon F \to G$ and $\psi : G \to F$ such that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. In other words, $F(A)$ is isomorphic to $G(A)$ via the two maps ψ_A and ϕ_A for all objects A in a functorial way, i.e., in such a way that the diagram (B.1) holds.

The functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ for any two given categories together with their natural transformations form a category.

The notion of equivalence of categories is important since it allows us to identify two categories which are apparently different.

Definition B.1.7. We say that two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there exist two functors $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ and $G\colon \mathcal{C}_2 \to \mathcal{C}_1$ such that $FG \cong \mathrm{id}_{\mathcal{C}_2}$, $GF \cong \mathrm{id}_{\mathcal{C}_1}$, where $\mathrm{id}_{\mathcal{C}}$ denotes the identity functor of a given category, defined in the obvious way, while $F \cong F'$ means that the two functors are isomorphic.

If F is a functor from the category $\mathcal{C}_1$ to the category $\mathcal{C}_2$ for any two objects A, $B \in \mathcal{C}_1$, by its very definition, F induces a function (that we denoted previously with F)

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}_1}(A, B) \to \mathrm{Hom}_{\mathcal{C}_2}(F(A), F(B)).$$

Definition B.1.8. Let F be a functor. We say that F is *faithful* if $F_{A,B}$ is injective, we say F is *full* if $F_{A,B}$ is surjective and we say that F is *fully faithful* if $F_{A,B}$ is bijective.

Next we want to formally define what it means for a functor to be *representable*. Let us first define the representation functors.

Definition B.1.9. Let $\mathcal{C}$ be a category, A a fixed object in $\mathcal{C}$. We define the two *representation functors* Hom^A, Hom_A as

$$\mathrm{Hom}^A\colon \mathcal{C}^{\mathrm{op}} \to (\text{sets}), \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(B, A),$$
$$\mathrm{Hom}_A\colon \mathcal{C} \to (\text{sets}), \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B),$$

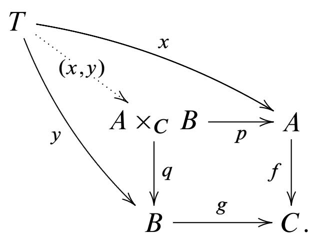
where (sets) denotes the category of sets. On the arrow $f \in \mathrm{Hom}(B, C)$ we have

$$\mathrm{Hom}^A(f)\phi = \phi \circ f,\ \phi \in \mathrm{Hom}^A(B), \quad \mathrm{Hom}_A(f)\psi = f \circ \psi,\ \psi \in \mathrm{Hom}_A(B).$$

Definition B.1.10. Let F be a functor from the category $\mathcal{C}$ to the category of sets. We say that F is *representable by* $X \in \mathcal{C}$ if for all $A \in \mathcal{C}$, $F \cong \mathrm{Hom}_A$ or $F \cong \mathrm{Hom}^A$.

We end our small exposition of categories by constructing the fibered product which is very important in our supergeometric constructions especially in Chapter 10.

Definition B.1.11. Given functors A, B, C from a category $\mathcal{C}$ to the category of (sets), and given natural transformations $f\colon A \to C$, $g\colon B \to C$, the *fibered product* $A \times_C B$ is the universal object making the following diagram commute:



One can see that

$$(A \times_C B)(R) = A(R) \times_{C(R)} B(R) = \{(a, b) \in A \times B \,|\, f(a) = g(b)\}.$$

If g is injective, that is, $g_R : B(R) \subset C(R)$, we have $(A \times_C B)(R) = f^{-1}(B(R))$.

The language of categories allows us to make (and prove) some sweeping generalizations about geometric objects without too much "forceful" computation. In particular, it also allows us to generalize the notion of a "point" to a T-point; this allows us to make more intuitive calculations with supergeometric objects. The main categories we discuss in this exposition are the categories of C^∞-supermanifolds, super Lie groups (a subcategory of C^∞-supermanifolds), superschemes, and superalgebraic groups.

B.2 Sheafification of a functor

In this section we discuss the concept of sheafification of a functor in supergeometry. Most of this material is known or easily derived from known facts. We include it here for completeness and lack of an appropriate reference.

Hereafter we shall make a distinction between a superspace X and its functor of points, that we shall denote by h_X or, if $X = \underline{\mathrm{Spec}}(A)$, by h_A.

We start by defining *local* and *sheaf* functors. For their definitions in the classical setting see for example [23], p. 16, or [29], Ch. VI.

Definition B.2.1. Let $F : \text{(salg)} \to \text{(sets)}$ be a functor. Fix $A \in \text{(salg)}$. Let $\{f_i\}_{i \in I} \subset A_0$, $(\{f_i\}_{i \in I}) = A_0$ and let $\phi_i : A \to A_{f_i}$, $\phi_{ij} : A_{f_i} \to A_{f_i f_j}$ be the natural morphisms, where $A_{f_i} := A[f_i^{-1}]$. We say that F is *local* if for any $A \in \text{(salg)}$ and for any family $\{\alpha_i\}_{i \in I}$, $\alpha_i \in F(A_{f_i})$ such that $F(\phi_{ij})(\alpha_i) = F(\phi_{ji})(\alpha_j)$ for all i and j, there exists a unique $\alpha \in F(A)$ with $F(\phi_i)(\alpha) = \alpha_i$ for all possible families $\{f_i\}_{i \in I}$ described above.

We want to rewrite this definition in more geometric terms in order to show that this is essentially the gluing property appearing in the usual definition of sheaf on a topological space.

We first observe that there is a contravariant equivalence of categories between the category of commutative superalgebras (salg) and the category of affine superschemes (aschemes), i.e., those superschemes that are the spectrum of some superalgebra (see Chapter 10 for more details). The equivalence is realized by $A \mapsto \underline{\mathrm{Spec}}(A)$ and it is explained in full details in Proposition 10.1.9. Hence a functor $F : \text{(salg)} \to \text{(sets)}$ can also be equivalently regarded as a functor $F : \text{(aschemes)}^{\mathrm{op}} \to \text{(sets)}$. With an abuse of notation we shall use the same letter to denote both functors.

Let F be a local functor, regarded as $F : \text{(aschemes)}^{\mathrm{op}} \to \text{(sets)}$, and let F_A be its restriction to the affine open subschemes of $\underline{\mathrm{Spec}}(A)$. Then F_A is a $\mathcal{B}$-sheaf in the usual sense (see Chapter 2); we must just forget the subscheme structure of the affine subschemes of $\underline{\mathrm{Spec}}(A)$ and treat them as open sets in the topological space $\mathrm{Spec}(A)$, their morphisms being the inclusions. Then F_A being a functor means that it is a presheaf in the Zariski topology, while the property detailed in Definition B.2.1

ensures the gluing of any family of local sections which agree on the intersection of any two parts of an open affine covering.

We now want to give the definition of local functor in a more general setting, so that it applies to the functor of points of supermanifolds and superschemes.

Definition B.2.2. Let $F: (\text{sspaces})^{\text{op}} \to (\text{sets})$ be a functor. We say that F is *local* or we also say it is a *sheaf* if it has the following property. For any superspace T and any open covering $\{T_i\}$ of T let $\phi_i: T_i \hookrightarrow T$, $\phi_{ij}: T_i \cup T_j \hookrightarrow T_i$ be the natural morphisms. If we have a family $\alpha_i \in F(T_i)$ such that $F(\phi_{ij})(\alpha_i) = F(\phi_{ji})(\alpha_j)$, then there exists a unique $\alpha \in F(T)$ with $F(\phi_i)(\alpha) = \alpha_i$.

One can readily check that this implies that, when F is restricted to the category of the open sets of a fixed superspace, F is a sheaf in the ordinary sense. As we already noticed, if we restrict F to the category of affine superschemes, this definition agrees with the previous one.

We leave the following proposition as an exercise to the reader. It is very similar to the proof in Chapter 10, Section 10.3.

Proposition B.2.3. *If X is a superspace, its functor of points is local.*

We now turn to the following problem. If we have a presheaf $\mathcal{F}$ on a topological space in the ordinary sense, we can always build its sheafification, which is a sheaf $\widetilde{\mathcal{F}}$ together with a sheaf morphism $\alpha: \mathcal{F} \to \widetilde{\mathcal{F}}$. This is the (unique) sheaf, which is locally isomorphic to the given presheaf and has the following universal property: any presheaf morphism $\phi: \mathcal{F} \to \mathcal{G}$, with $\mathcal{G}$ a sheaf, factors via α (for more details on this construction, see Chapter 2, Section 2.2). We now want to give the same construction in our more general setting.

The existence of sheafification of a functor from the category of algebras to the category of sets is granted in the ordinary case by [23], Ch. I, §1, no. 4, which is also nicely summarized in [23], Ch. III, §1, no. 3. The proof is quite formal and one can carry it to the supergeometric setting. We however prefer to introduce Grothendieck topologies and the concept of *site* and to construct the sheafification of a functor through them. In fact, as we shall see, very remarkably Grothendieck's treatment is far more general and it comprehends supergeometry. For more details we refer the reader to the classical account by Grothendieck [38], [41] and the more modern treatment by Vistoli [77].

Definition B.2.4. We call a category $\mathcal{C}$ a *site* if it has a *Grothendieck topology*, i.e., to every object $U \in \mathcal{C}$ we associate a collection of so-called *coverings* of U, that is, sets of arrows $\{U_i \to U\}$ such that the following holds:

(1) If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.

(2) If $\{U_i \to U\}$ is a covering and $V \to U$ is any arrow, then the fibered products $\{U_i \times_U V\}$ exist and the collection of projections $\{U_i \times_U V \to V\}$ is a covering.

(3) If $\{U_i \to V\}$ is a covering and for each index i we have a covering $\{V_{ij} \to U_i\}$, then the collection $\{V_{ij} \to U_i \to U\}$ is a covering of U.

The category of superschemes and its subcategory of affine superschemes are sites, by taking for each object the collection of its (affine) coverings, as in the ordinary setting. Similarly also the category of commutative superalgebras is also a site (for the existence of fibered products in such categories and for more details on coverings see Sections 10.3, 9.4). Such Grothendieck topologies, with an abuse of terminology, are commonly referred to as *Zariski topologies* (one should prove that all of these topologies are essentially equivalent). We shall not dwell upon the technicalities of Grothendieck topologies, referring the reader to [77] where all the many subtleties are discussed in the fullest detail.

Definition B.2.5. Let $\mathcal{C}$ be a site. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\text{sets})$ is called a *sheaf* if for all objects $U \in \mathcal{C}$, coverings $\{U_i \to U\}$ and families $a_i \in F(U_i)$, we have the following. Let $p^1_{ij}\colon U_i \times_U U_j \to U_i$, $p^2_{ij}\colon U_i \times_U U_j \to U_j$ denote the natural projections and assume that $F(p^1_{ij})(a_i) = F(p^2_{ij})(a_j) \in F(U_i \times_U U_j)$ for all i, j. Then there exists a unique $a \in F(U)$ whose pullback to $F(U_i)$ is a_i for every i.

Again one can check that the functor of points of superschemes and supermanifolds are sheaves in this more general setting for the corresponding sites.

We are ready for the sheafification of a functor in this very general setting.

Definition B.2.6. Let $\mathcal{C}$ be a site and let $F\colon \mathcal{C}^{\mathrm{op}} \to (\text{sets})$ be a functor. A *sheafification* of F is a sheaf $\widetilde{F}\colon \mathcal{C}^{\mathrm{op}} \to (\text{sets})$ with a natural transformation $\alpha\colon F \to \widetilde{F}$ such that:

(1) For any $U \in \mathcal{C}$ and $\xi, \eta \in F(U)$ such that $\alpha_U(\xi) = \alpha_U(\eta)$ in $\widetilde{F}(U)$, there is a covering $\{\sigma_i\colon U_i \to U\}$ with $F(\sigma_i)(\xi) = F(\sigma_i)(\eta)$ in $F(U_i)$.

(2) For any $U \in \mathcal{C}$ and any $\xi \in \widetilde{F}(U)$, there is a covering $\{\sigma_i\colon U_i \to U\}$ and elements $\xi_i \in F(U_i)$ such that $\alpha_{U_i}(\xi_i) = \widetilde{F}(\sigma_i)(\xi)$ in $\widetilde{F}(U_i)$.

The next theorem states the fundamental properties of the sheafification.

Theorem B.2.7 ([77], p. 42). *Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{op}} \to (\text{sets})$ a functor.*

(1) *If $\widetilde{F}$ is a sheafification of F with $\alpha\colon F \to \widetilde{F}$, then any morphism $\psi\colon F \to G$, with G a sheaf, factors uniquely through $\widetilde{F}$.*

(2) *F admits a sheafification $\widetilde{F}$, unique up to a canonical isomorphism.*

We shall use this construction for the following supergeometric categories: $\mathcal{C} = (\text{smflds})$, $\mathcal{C} = (\text{sschemes})$, $\mathcal{C} = (\text{aschemes})$, or equivalently $\mathcal{C}^{\mathrm{op}} = (\text{salg})$.

Observation B.2.8. Let $F\colon \mathcal{C}^{\mathrm{op}} \to (\text{sets})$ be a functor, $\widetilde{F}$ its sheafification, where $\mathcal{C}$ is one of the supergeometric categories specified above. Let A be an object of $\mathcal{C}$ and F_A the restriction of the functor F to the category of the open subobjects of A (see above). Then $\widetilde{F_A}$ is the sheafification of F_A in the usual sense, that is, the sheafification of F as a sheaf defined on the topological space underlying A. In particular, since a sheaf and its sheafification are locally isomorphic, we have $F_{A,p} \cong \widetilde{F}_{A,p}$, i.e., they have isomorphic stalks (via the natural map $\alpha\colon F \to \widetilde{F}$) at any point p and for all objects A. To simplify the notation we shall drop the suffix A and write just F_p instead of $F_{A,p}$.

Proposition B.2.9. *Let $F, G\colon$ (sspaces) $\to$ (sets) be local functors and let $\alpha\colon F \to G$ be a natural transformation. Assume that $F_A \cong G_A$ via α, where F_A and G_A denote the ordinary sheaves corresponding to the restrictions of F and G to the category of open subspaces in A (morphisms given by the inclusions). Then α is an isomorphism, hence $F \cong G$.*

Proof. We can certainly write an inverse for α_A for every object A, the problem is to see if it is well behaved on the arrows. However, this is true because α is a natural transformation. □

The rest of this section is devoted to proving the following result.

Theorem B.2.10. *Let $F, G\colon$ (salg) $\to$ (sets) be two functors, with G a sheaf. Assume that we have a natural transformation $F \to G$, which is an isomorphism on local superalgebras, i.e., $F(R) \cong G(R)$ (via this map) for all local superalgebras R. Then $\widetilde{F} \cong G$. In particular, $F \cong G$ if also F is a sheaf.*

Lemma B.2.11. *Let $F\colon$ (salg) $\to$ (sets) be a functor. For $p \in \mathrm{Spec}(A)$, let $F_p = \varinjlim F(R)$, where the direct limit is taken for the rings R corresponding to the open affine subschemes of $\underline{\mathrm{Spec}}(A)$ containing p. Then $F_p = F(A_p)$.*

Proof. By Yoneda's lemma, we have

$$F_p = \varinjlim F(R) = \varinjlim \mathrm{Hom}(h_R, F) = \mathrm{Hom}(h_{\varinjlim R}, F) = \mathrm{Hom}(h_{A_p}, F) = F(A_p)$$

as $\varinjlim$ and Hom commute (see [51], p. 141) and $A_p = \varinjlim R$ (see [2], p. 47). □

Lemma B.2.12. *Let $A \in$ (salg), $p \in \mathrm{Spec}(A_0)$. Then A_p (the localization at p of A as an A_0-module) is a local superalgebra, whose maximal ideal is $\mathfrak{m} = (\mathfrak{m}_0, (A_1)_p)$, where $\mathfrak{m}_0$ is the maximal ideal in the algebra $(A_0)_p = (A_p)_0$.*

Proof. From $A = A_0 \oplus A_1$ we get $A_p = (A_0)_p \oplus (A_1)_p$, and clearly this is a superalgebra with $(A_p)_0 = (A_0)_p$, $(A_p)_1 = (A_1)_p$. Now let us consider $\mathfrak{m} := \big(\mathfrak{m}_0, (A_1)_p\big) = \mathfrak{m}_0 + (A_1)_p$. By the above, $\mathfrak{m} \neq A_p = (A_0)_p \oplus (A_1)_p$. Take $x \notin \mathfrak{m}$: then $x = x_0 + x_1$ with $x_0 \in (A_0)_p$, $x_1 \in (A_1)_p$, so x_0 is invertible in $(A_0)_p \subseteq (A_1)_p$ and x_1 is nilpotent, hence x is invertible. □

We are ready for the proof of Theorem B.2.10:

Proof of Theorem B.2.10. Assume first that F and G are local. Since $F(R) \cong G(R)$ for all local algebras R, by Lemma B.2.11 this implies that $F_p \cong G_p$ for all $p \in \mathrm{Spec}(A)$ and all superalgebras A. Hence $F_A \cong G_A$ by [43], Ch. II, Section 1.1. By Proposition B.2.9, we have $F \cong G$ (all isomorphisms have to be intended via the natural transformation $\alpha\colon F \to \widetilde{F}$).

Now assume that F is not a sheaf. We have $\alpha\colon F \to \widetilde{F} \to G$ by Theorem B.2.7. If $A \in$ (salg), then by restricting our functors to the open affine sets in $\underline{\mathrm{Spec}}(A)$ we get

$F_A \to \widetilde{F}_A \to G_A$. By Observation B.2.8, F_A and $\widetilde{F}_A$ are locally isomorphic via α, so $F_p \cong \widetilde{F}_{A,p}$. By hypothesis, $F(R) \cong G(R)$, so $F_p \cong G_p$ by Lemma B.2.11, hence $\widetilde{F}_p \cong G_p$. Arguing as before, we obtain the result. □

Along the same lines, the reader can prove the following proposition:

Proposition B.2.13. *Let $\phi\colon F \to G$ be a natural transformation between two local functors from* (salg) *to* (sets). *Assume that we know ϕ_R for all local superalgebras R. Then ϕ is uniquely determined.*

B.3 Super Nakayama's lemma and projective modules

Let A be a commutative superalgebra.

Definition B.3.1. A *projective* A-module M is a direct summand of $A^{m|n}$. In other words, it is a projective module in the classical sense respecting the grading: $M_0 \subset A_0^{m|n}$, $M_1 \subset A_1^{m|n}$.

Observation B.3.2. As in the classical setting, being projective is equivalent to the exactness of the functor $\mathrm{Hom}(M, \cdot)$.

We want to show that a projective A-module has the property of being locally free, that is its localization M_p into primes p of A_0 is free as an A_p-module. This result allows us to define the *rank* of a projective module as it happens in the ordinary case.

We start with a generalization of Nakayama's lemma.

Lemma B.3.3 (Super Nakayama lemma). *Let A be a local supercommutative ring with maximal homogeneous ideal $\mathfrak{m}$. Let E be a finitely generated module for the ungraded ring A.*

(1) *If $\mathfrak{m}E = E$, then $E = 0$; more generally, if H is a submodule of E such that $E = \mathfrak{m}E + H$, then $E = H$.*

(2) *Let $(v_i)_{1 \le i \le p}$ be a basis for the k-vector space $E/\mathfrak{m}E$ where $k = A/\mathfrak{m}$. Let $e_i \in E$ be above v_i. Then the e_i generate E. If E is a supermodule for the super ring A, and v_i are homogeneous elements of the super vector space $E/\mathfrak{m}E$, we can choose the e_i to be homogeneous too (and hence of the same parity as the v_i).*

(3) *Suppose that E is projective, i.e., there is an A-module F such that $E \oplus F = A^N$, where A^N is the free module for the ungraded ring A of rank N. Then E (and hence F) is free, and the e_i above form a basis for E.*

Proof. The proofs are easy extensions of the ones in the commutative case. We begin the proof of (1) with the following observation: if B is a commutative local ring with $\mathfrak{n}$ a maximal ideal, then a square matrix R over B is invertible if and only if it is invertible modulo $\mathfrak{n}$ over the field $B/\mathfrak{n}$. In fact if this is so, $\det(R) \notin \mathfrak{n}$ and so is a unit of B.

This said, let u_i, $(1 < i < N)$ generate E. If $E = \mathfrak{m}E$, we can find $m_{ij} \in \mathfrak{m}$ so that $u_i = \sum_j m_{ij} u_j$ for all i. Hence, if L is the matrix with entries $\delta_{ij} - m_{ij}$, then

$$L \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = 0.$$

It is now enough to prove that L has a left inverse P. Then multiplying the above from the left by P, we get $u_i = 0$ for all i and so $E = 0$. It is even true that L is invertible. To prove this, let us consider $B = A/J_A$ where J_A is the ideal generated by A_1. Since $J_A \subset \mathfrak{m}$ we have

$$A \to B = A/J_A \to k = A/\mathfrak{m}.$$

Let L_B (resp. L_k) be the reduction of L modulo J_A (respectively modulo $\mathfrak{m}$). Then B is local and its maximal ideal is $\mathfrak{m}/J_A$, where L_k is the reduction of L_B mod $\mathfrak{m}/J_A$. But B is commutative and $L_k = I$, and so L_B is invertible. But then L is invertible. If more generally we have $E = H + \mathfrak{m}E$, then $E/H = \mathfrak{m}(E/H)$ and so $E/H = 0$, which is to say that $E = H$.

To prove (2), let H be the submodule of E generated by the e_i. Then $E = \mathfrak{m}E + H$ and so $E = H$.

We now prove (3). Clearly F is also finitely generated. We have $k^N = A^N/\mathfrak{m}^N = E/\mathfrak{m}E \oplus F/\mathfrak{m}F$. Let (w_j) be a basis of $F/\mathfrak{m}F$ and let f_j be elements of F above w_j. Then by (2), the e_i, f_j form a basis of A^N, while the e_i (resp. f_j) generate E (resp. F). Now there are exactly N of the e_i, f_j, and so if X denotes the $(N \times N)$-matrix with columns $e_1, \dots, f_1, \dots$, then for some $(N \times N)$-matrix Y over A we have $XY = I$. Hence $X_B Y_B = I$ where the suffix "B" denotes reduction modulo B. However, B is commutative and so $Y_B X_B = I$. Thus X has a left inverse over A, which must be Y so that $YX = I$. If there is a linear relation among the e_i and the f_j, and if x is the column vector whose components are the coefficients of this relation, then $Xx = 0$. But this implies that $x = YXx = 0$. In particular E is a free module with basis (e_i). □

We now wish to give a characterization of projective modules.

Theorem B.3.4. *Let M be a finitely generated A-module, with A finitely generated over A_0 where A_0 is noetherian. Then:*

(1) *M is projective if and only if M_p is free for all p primes in A_0.*

(2) *M is projective if and only if $M[f_i^{-1}]$ is free for all f_i's such that $(f_1, \dots, f_r) = A_0$.*

Proof. (1) If M is projective, by part (3) of Nakayama's lemma, we have that M_p is free since it is a module over the supercommutative ring A_p.

Now assume that M_p is free for all primes $p \in A_0$. Recall that

$$\mathrm{Hom}_{A[U^{-1}]}(M[U^{-1}], N[U^{-1}]) = \mathrm{Hom}_A(M, N)[U^{-1}]$$

for U a multiplicatively closed set in A_0. Recall also that given A_0-modules N, N', N'', we have that $0 \to N' \to N \to N''$ is exact if and only if $0 \to N'_p \to N_p \to N''_p$ is exact for all primes p in A_0. So given an exact sequence $0 \to N' \to N \to N''$, since M_p is free, we obtain the exact sequence

$$0 \to \mathrm{Hom}(M_p, N'_p) \to \mathrm{Hom}(M_p, N_p) \to \mathrm{Hom}(M_p, N''_p) \to 0$$

for all the primes p. Hence by the previous observation,

$$0 \to \mathrm{Hom}(M, N')_p \to \mathrm{Hom}(M, N)_p \to \mathrm{Hom}(M, N'')_p \to 0$$

and

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to 0.$$

Thus M is projective.

(2) That M_p is free for all primes p is equivalent to $M[f_i^{-1}]$ being free for $(f_1, \dots, f_r) = A_0$ is a standard fact of commutative algebra and can be found in [28], p. 623, for example. □

Definition B.3.5. Let M be a finitely generated projective A-module. We say that M has *rank* $r|s$ if $M_p \cong A^{r|s}$ for all primes p in A. In the light of the previous proposition one can show that the rank is locally constant, that is, if $M_p \cong A^{r|s}$, then there exists an open neighbourhood U of $p \in \mathrm{Spec}\, A$ for which $M_{p'} = A^{r|s}$, $p' \in U$.

Remark B.3.6. As in the ordinary setting we have a correspondence between projective A-modules and locally free sheaves on $\mathrm{Spec}\, A_0$. In this correspondence, given a projective A-module M, we view M as an A_0-module and build a sheaf of modules $\mathcal{O}_M$ on A_0. The global sections of this sheaf are isomorphic to M itself, and locally, i.e., on the open sets $U_{f_i} = \{p \in \mathrm{Spec}\, A_0 | (f_i) \not\subset p\}$, $f_i \in A_0$,

$$\mathcal{O}_M(U_{f_i}) = M[f_i^{-1}].$$

More details on this construction in the ordinary setting can be found, for example, in [43], Ch. II. As for its generalization to the super context, it is straightforward.

B.4 References

For a complete introduction to categories we refer the reader to [55], however a good review of the main properties can also be found in [51]. As for the sheafification of a functor we refer to [77] and of course to [38] by Grothendieck. Finally for the super Nakayama lemma see [76] and for the ordinary setting [57].

C

Fréchet superspaces

The purpose of this appendix is to provide a quick introduction to Fréchet spaces, superspaces and supersheaves, that is crucial for the understanding of a few key supergeometric facts detailed in Chapter 4, Section 4.5. We first give an overview of the ordinary setting, by providing the definition of Fréchet space and summarizing the main results of the theory. It is of course impossible to give a fair account of this subject in a few pages and for this reason we have provided precise references. We then proceed to describe in more detail the theory of Fréchet superspaces and supersheaves.

C.1 Fréchet spaces

We want to start with the classical definition of Fréchet space. We first recall some preliminary topology definitions, referring the reader to [64], Ch. I, for all the results and further reading on these topics.

Let $\mathbb{R}$ be our ground field.

Definition C.1.1. A *topological vector space* V is both a topological space and a vector space such that the vector space operations are continuous. A topological vector space is *locally convex* if its topology admits a basis consisting of convex sets (a set A is *convex* if $(1-t)x + ty \in A$ for all $x, y \in A$ and $t \in [0, 1]$).

We say that a locally convex topological vector space is a *Fréchet space* if its topology is induced by a translation-invariant metric d and the space is complete with respect to d, that is, all the Cauchy sequences are convergent.

There is an important characterization of Fréchet spaces that we are going to need in the sequel. Let us start with the definition of seminorm and a proposition relating seminorms and a translation-invariant metric.

Definition C.1.2. A *seminorm* on a vector space V is a real-valued function p such that for all $x, y \in V$ and scalars a we have:

(1) $p(x + y) \leq p(x) + p(y)$,

(2) $p(ax) = |a|p(x)$,

(3) $p(x) \geq 0$.

Notice that the difference between the norm and the seminorm comes from the last property: we do not ask that if $x \neq 0$, then $p(x) > 0$, as we would do for a norm.

Proposition C.1.3. *If $\{p_i\}_{\{i\in\mathbb{N}\}}$ is a countable family of seminorms on a topological vector space V, separating points (that is, if $x \neq 0$, there is an i with $p_i(x) \neq 0$), then there exists a translation-invariant metric d inducing the topology, defined in terms of the $\{p_i\}$:*

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)}.$$

Proof. See [64], p. 27. □

The next proposition characterizes Fréchet spaces, giving an effective method to construct them using seminorms.

Proposition C.1.4. *A topological vector space V is a Fréchet space if and only if it satisfies the following three properties:*

- *it is complete as a topological vector space;*
- *it is a Hausdorff space;*
- *its topology is induced by a countable family of seminorms $\{p_i\}_{i\in\mathbb{N}}$, i.e., $U \subset V$ is open if and only if for every $u \in U$ there exists $K \geq 0$ and $\varepsilon > 0$ such that $\{v \mid p_k(u-v) < \varepsilon$ for all $k \leq K\} \subset U$.*

Definition C.1.5. We say that a sequence (x_n) in V *converges to x* in the Fréchet space topology defined by a family of seminorms if and only if it converges to x with respect to each of the given seminorms. In other words, $x_n \to x$ if and only if $p_i(x_n - x) \to 0$ for each i.

Two families of seminorms defined on the locally convex vector space V are said to be *equivalent* if they induce the same topology on V.

To construct a Fréchet space, one typically starts with a locally convex topological vector space V and defines a countable family of seminorms $\{p_k\}$ on V inducing its topology and such that:

(I) if $x \in V$ and $p_k(x) = 0$ for all $k \geq 0$, then $x = 0$ (*separation property*);

(II) if (x_n) is a sequence in V which is Cauchy with respect to each seminorm, then there exists $x \in V$ such that (x_n) converges to x with respect to each seminorm (*completeness property*).

The topology induced by these seminorms (as explained above) turns V into a Fréchet space; property (I) ensures that it is Hausdorff, while the property (II) guarantees that it is complete. A translation-invariant complete metric inducing the topology on V can then be defined as above.

The most important example of Fréchet space, at least for the application we have in mind, is the vector space $C^\infty(U)$, the space of smooth functions on the open set $U \subseteq \mathbb{R}^n$ or more generally the vector space $C^\infty(M)$, where M is a differentiable manifold. For each open set $U \subset \mathbb{R}^n$ (or $U \subset M$), for each $K \subset U$ compact and for each multi-index I, we define

$$\|f\|_{K,I} := \sup_{x \in K} \left| \left(\frac{\partial^{|I|}}{\partial x^I}(f) \right)(x) \right|, \quad f \in C^\infty(U).$$

Each $\|\cdot\|_{K,I}$ defines a seminorm. The family of seminorms obtained by considering all of the multi-indices I and the (countable number of) compact subsets K covering U satisfies the properties (I) and (II) detailed above, hence makes $C^\infty(U)$ into a Fréchet space. The sets of the form

$$\{f \in C^\infty(U) \mid \|f - g\|_{K,I} < \varepsilon\}$$

with fixed $g \in C^\infty(U)$, $K \subseteq U$ compact, and multi-index I are open sets and together with their finite intersections form a basis for the topology.

All these constructions and results can be generalized to smooth manifolds, as we now briefly outline.

Let M be a smooth manifold and let U be an open subset of M. If K is a compact subset of U and D is a differential operator over U, then

$$p_{K,D}(f) := \sup_{x \in K} |D(f)|$$

is a seminorm. The family of all the seminorms $p_{K,D}$ with K and D varying among all compact subsets and differential operators respectively is a separating family of seminorms endowing $C^\infty_M(U)$ with the structure of a complete locally convex vector space. Moreover there exists an equivalent countable family of seminorms, hence $C^\infty_M(U)$ is a Fréchet space. Let indeed $\{V_j\}$ be a countable open cover of U by open coordinate subsets, and let, for each j, $\{K_{j,i}\}$ be a countable family of compact subsets of V_j such that $\bigcup_i K_{j,i} = V_j$. We have the countable family of seminorms

$$p_{K,I} := \sup_{x \in K} \left| \left(\frac{\partial^{|I|}}{\partial x^I}(f) \right)(x) \right|, \quad K \in \{K_{j,i}\},$$

inducing the topology. Notice that $C^\infty_M(U)$ is also an algebra: the product of two smooth functions being a smooth function. It is thus natural in this context to introduce the notion of Fréchet algebra.

Definition C.1.6. A Fréchet space V is said to be a *Fréchet algebra* if its topology can be defined by a countable family of submultiplicative seminorms, i.e., a countable family $\{q_i\}_{i \in \mathbb{N}}$ of seminorms satisfying

$$q_i(fg) \leq q_i(f) q_i(g) \quad \text{for all } i \in \mathbb{N}.$$

If we come back to our prototype of Fréchet space $C^\infty(\mathbb{R}^n)$, one can check that it is also a Fréchet algebra, the countable family of submultiplicative seminorms given by

$$q_{K_{i,m},j}(f) := 2^j \sup_{x \in K_{i,m}, |I| \le j} \left| \left(\frac{\partial^{|I|}}{\partial x^I}(f) \right)(x) \right|.$$

We end this section with a definition of a Fréchet sheaf.

Definition C.1.7. Let $\mathcal{F}$ be a sheaf of real vector spaces over a manifold M. $\mathcal{F}$ is a *Fréchet sheaf* if:

(1) for each open set $U \subseteq M$, $\mathcal{F}(U)$ is a Fréchet space;

(2) for each open set $U \subseteq M$ and for each open cover $\{U_i\}$ of U, the topology of $\mathcal{F}(U)$ is the initial topology with respect to the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_i)$, that is, the coarsest topology making the restriction morphisms continuous.

As a consequence, we have that each restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ ($V \subseteq U$) is continuous.

A morphism of sheaves $\psi\colon \mathcal{F} \to \mathcal{F}'$ is said to be *continuous* if the map $\mathcal{F}(U) \to \mathcal{F}'(U)$ is continuous for each open subset $U \subseteq M$.

C.2 Fréchet superspaces

We are now ready to introduce the notion of Fréchet space in the super context.

Definition C.2.1. (1) We say that a super vector space $V = V_0 \oplus V_1$ is a *super Fréchet space* if there exist two families of homogeneous seminorms $\{p_{0i}\}$ and $\{p_{1i}\}$ defined on V_0 and V_1, respectively, with respect to which V_0 and V_1 are Fréchet spaces. We will usually denote by p_i a generic seminorm from one of the two families.

(2) Let $A = A_0 \oplus A_1$ be a super Fréchet space with respect to the family of seminorms $\{p_i\}_{i \in I}$. Suppose also that A is a superalgebra with multiplication m. We say that A is a *super Fréchet algebra* if the topology is defined by an equivalent family of submultiplicative seminorms $\{q_i\}$: $q_i(ab) \le q_i(a)q_i(b)$.

(3) Suppose now that $\mathcal{F}$ is a sheaf such that $\mathcal{F}(U)$ is a super Fréchet algebra for each U. We say that $\mathcal{F}$ is a *super Fréchet sheaf* if for each open set U and for any open cover $\{U_i\}$ of U the topology of $\mathcal{F}(U)$ is the initial topology with respect to the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_i)$.

We now turn to the example most interesting to us, namely the superalgebra of sections on a supermanifold.

Let $M = (|M|, \mathcal{O}_M)$ be a supermanifold. Fix now an open subset $U \subseteq |M|$. For each compact subset $K \subset U$ and each differential operator D over U, define

$$p_{K,D}(f) := \sup_{x \in K} |\widetilde{(D(f))}(x)|, \quad f \in \mathcal{O}_M(U).$$

As before, one can readily check that each $p_{K,D}$ defines a seminorm. The family of seminorms obtained by considering all the differential operators and the compact subsets K covering U endows $\mathcal{O}_M(U)$ with a Hausdorff locally convex topology (as before), where the open sets that form a basis for the topology are

$$\{f \in \mathcal{O}_M(U) \mid p_{K,D}(f-g) < \varepsilon\},$$

with fixed $g \in \mathcal{O}_M(U)$, $K \subseteq U$ compact, $D \in \mathrm{Diff}(U)$, the differential operators on U, and $\varepsilon > 0$, together with their finite intersections.

In complete analogy with the ordinary setting one can prove the following proposition.

Proposition C.2.2. *$\mathcal{O}_M$ is a super Fréchet sheaf.*

Before proving it we need some preliminary results that we collect in the next lemma and proposition.

Lemma C.2.3. *Let U be a chart with coordinates t^i, θ^j and let $\{f_n\}$ be a sequence in $\mathcal{O}_M(U)$:*

$$f_n = \sum_I f_{nI}\theta^I, \quad f_{nI} \in C^\infty_M(U).$$

(1) *$\{f_n\}$ is a Cauchy sequence in $\mathcal{O}_M(U)$ if and only if the sequence $\{f_{nI}\}$ is a Cauchy sequence in $C^\infty(U)$ for each I.*

(2) *$\mathcal{O}_M(U)$ is complete, moreover it is a super Fréchet algebra.*

Proof. (1) Let K be a compact subset of U and let $D = \sum a_{\gamma,I}\frac{\partial^{|\gamma|+|I|}}{\partial t^\gamma \theta^I}$ be a super differential operator. We have

$$\begin{aligned} p_{K,D}\Big(\sum_J f_J\theta^J\Big) &= \sup_{x\in K}\Big|\widetilde{\Big(\sum_{\gamma,I} a_{\gamma,I}\frac{\partial}{\partial t^\gamma \partial\theta^I}\sum_J f_J\theta^J\Big)}(x)\Big| \\ &\le \sum_{\gamma,I}\max_{x\in K}(a_{\gamma,I})\,p_{K,\frac{\partial}{\partial t^\gamma}}(f_I), \end{aligned}$$

which finishes the proof of (1).

(2) It is well known that $C^\infty_M(U)$ is complete with respect to such a locally convex topology (see, for instance, [73]).

In order to show that $\mathcal{O}_M(U)$ is a Fréchet space it is then enough to produce a countable family of seminorms satisfying the properties detailed above. This is obtained, as in the classical case, by considering the family $\{p_{K_n,\frac{\partial}{\partial t^\gamma\partial\theta^I}}\}_{n,\gamma,I}$, where each K_n is a family of compact sets in U such that K_n is contained in the interior of K_{n+1} and $\bigcup K_n = U$ (see [16], [26]).

We now show that $\mathcal{O}_M(U)$ is a Fréchet superalgebra. This follows considering the family of seminorms

$$q_{\alpha,K_n} := 2^{\alpha+2n}\max_{|\gamma|\le\alpha,|I|}\Big(p_{K_n,\frac{\partial}{\partial t^\gamma\partial\theta^I}}\Big), \quad \alpha \in \mathbb{N}.$$

It is not difficult to check that the family $\{q_{\alpha,K_n}\}_{\alpha,n}$ is equivalent to the family $\{p_{K_n,\frac{\partial}{\partial t^\gamma \partial \theta^I}}\}_{n,\gamma,I}$ and that they are submultiplicative. □

Proposition C.2.4. *Let $\{U_i\}$ be an open cover of an open set $U \subset |M|$, let also $\{s_n\}$ be a sequence in $\mathcal{O}_M(U)$.*

(1) *$\{s_n\}$ converges to s in $\mathcal{O}_M(U)$ if and only if $\{s_n|_{U_i}\}$ converges to $s|_{U_i}$ in $\mathcal{O}_M(U_i)$ for each U_i.*

(2) *$\{s_n\}$ is a Cauchy sequence in $\mathcal{O}_M(U)$ if and only if $\{s_n|_{U_i}\}$ is a Cauchy sequence in $\mathcal{O}_M(U_i)$ for each U_i.*

(3) *$\mathcal{O}_M(U)$ is complete for each open subset U.*

(4) *$\mathcal{O}_M(U)$ is a super Fréchet algebra for each open subset U.*

Proof. (1) Clearly if $\{s_n\} \to 0$, then $s_n|_{U_i} \to 0$ for each i. Suppose vice versa that for each integer i, $s_n|_{U_i} \to 0$ and let K be a compact subset of U. There exists a finite open cover V_j of K such that each $\bar{V}_j$ is compact and for each j there exists i such that $V_j \subset U_i$ (see [26]). Then (1) follows from the inequality

$$p_{K,D}(s_n) \leq \sum_j p_{K\cap\bar{V}_j, D|_{K\cap\bar{V}_j}}(s_n).$$

(2) If $\{s_n\}$ is a Cauchy sequence in $\mathcal{O}_M(U)$ and $\{U_i\}$ are charts covering U, then there exists a family $\{s_{U_i}\}$ such that $s_n|_{U_i}$ converges to s_{U_i} for each i. An easy check shows that the various s_{U_i} glue together and define a section s in $\mathcal{O}_M(U)$. Clearly s_n converges to s.

(3) The fact that $\mathcal{O}_M(U)$ is complete follows by Lemma C.2.3, by considering a countable open cover $\{U_i\}$ of U by open supercharts and points (1) and (2).

(4) The fact that $\mathcal{O}_M(U)$ is a super Fréchet algebra follows by considering the family of seminorms

$$q_{\alpha,K_{i,n}} := 2^\alpha \max_{|\gamma|\leq\alpha,|I|} (p_{K_{i,n},\frac{\partial}{\partial t^\gamma \partial\theta^I}}),$$

where $\{K_{i,n}\}_{n\in\mathbb{N}}$ is a countable family of compact subsets covering U_i. □

Proof of Proposition C.2.2. The fact that $\mathcal{O}_M(U)$ is a Fréchet algebra for each open subset U is the content of the previous proposition. The fact that the topology of $\mathcal{O}_M(U)$ is the initial topology induced by the restriction maps is the content of item (1) of Proposition C.2.4. □

With such a topology, all the important geometrical operations that can be performed on the sheaf turn out to be continuous. We have in fact the following proposition.

Proposition C.2.5. (1) *If D is a super differential operator on M, then $D\colon \mathcal{O}_M(U) \to \mathcal{O}_M(U)$ is continuous for each U.*

(2) *If $\psi\colon M \to N$ is a supermanifold morphism, then $\psi^*\colon \mathcal{O}(N) \to \mathcal{O}(M)$ is continuous.*

Proof. (1) is clear because of the way the seminorms, hence the topology, are defined.

(2) Due to Proposition C.2.4 we can suppose that M and N are superdomains with coordinates t^i, θ^j and x^r, ξ^s, respectively. We have to prove that if $\{f_n\}$ is a sequence in $\mathcal{O}(N)$ converging to zero, then the sequence $\{\psi^*(f_n)\}$ converges to zero in $\mathcal{O}(M)$, i.e., for all compact subsets K and all differential operators $D = \frac{\partial^{|\alpha|+|I|}}{\partial t^\alpha \partial \theta^I}$, $p_{D,K}(\psi^*(f_n))$ tends to zero. Let $a \in \mathrm{Diff}_0(M)$, the differential operators of degree zero, namely $a \in \mathcal{O}(M)$. Then

$$p_{a,K}(\psi^*(f_n)) = \sup_{x \in K} |\widetilde{(a \cdot \psi^*(f_n))(x)}| \tag{C.1}$$

$$\leq (\sup_{x \in K} |a(x)|)(\sup_{x \in K} |f_n(|\psi|(x))|), \tag{C.2}$$

and the result follows easily. If $D = \frac{\partial}{\partial y}$ with y denoting indifferently an even or odd variable, then using the chain rule (4.4), we have

$$p_{D,K}(\psi^*(f_n)) := \sup_{x \in K} \left(\frac{\partial \psi^*(x^r)}{\partial y} \psi^* \frac{\partial f_n}{\partial x^r} \right)$$

and the result follows from equation (C.1). The case of a differential operator $D = \frac{\partial^{|\alpha|}}{\partial t^\alpha}$ is treated similarly. □

Bibliography

The numbers at the end of each item refer to the pages on which the respective work is cited.

[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis and applications*. Appl. Math. Sci. 75, Springer-Verlag, New York 1993. 102

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass., 1969. 44, 77, 197, 265

[3] Yu. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, and M. V. Zaicev, *Infinite dimensional Lie superalgebras*. De Gruyter Exp. Math. 7, Walter de Gruyter, Berlin 1992. 9

[4] L. Balduzzi, C. Carmeli, and G. Cassinelli, Super G-spaces. In *Symmetry in mathematics and physics*, Contemp. Math. 490, Amer. Math. Soc., Providence, RI, 2009, 159–176. 153, 172

[5] L. Balduzzi, C. Carmeli, and R. Fioresi, Quotients in supergeometry. In *Symmetry in mathematics and physics*, Contemp. Math. 490, Amer. Math. Soc., Providence, RI, 2009, 177–187. 172

[6] C. Bartocci, U. Bruzzo, and D. Hernandez-Ruiperez, *The geometry of supermanifolds*. Math. Appl. 71, Kluwer Academic Publishers, Dordrecht 1991. 89

[7] M. Batchelor, Graded manifolds and vector bundles: A functorial correspondence. *J. Math. Phys.* **26** (1985), 1578–1582. 63

[8] M. Batchelor, Two approaches to supermanifolds. *Trans. Amer. Math. Soc.* **258** (1980), 257–270.

[9] M. Batchelor, The structure of supermanifolds. *Trans. Amer. Math. Soc.* **253** (1979), 329–338.

[10] F. A. Berezin, *Introduction to superanalysis*. Math. Phys. Appl. Math. 9, D. Reidel Publishing C., Dordrecht 1987. viii, 15, 27, 53, 89

[11] F. A. Berezin and D. A. Leites, Supermanifolds. *Dokl. Akad. Nauk SSSR* **224** (1975), 505–508. 89, 140

[12] G. Bergman, The diamond lemma for ring theory. *Adv. Math.* **29** (1978), 178–218. 24

[13] A. Borel, *Linear algebraic groups*. Grad. Texts in Math. 126, Springer-Verlag, New York 1991.

[14] H. Boseck, Affine Lie supergroups. *Math. Nachr.* **143** (1989), 303–327. 230

[15] H. Boseck, Classical Lie supergroups. *Math. Nachr.* **148** (1990), 81–115. 230

[16] N. Bourbaki, *General topology*. Chapters 1-4, Elem. Math. (Berlin), Springer-Verlag, Berlin 1989. 273

[17] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIII. Variétés différentielles et analytiques.* Fascicule de résultats (Paragraphes 1 à 7), Actualités Scientifiques et Industrielles 1333, Hermann, Paris 1967. 82

[18] N. Bourbaki, *Lie groups and Lie algebras*. Chapters 1–3, Elem. Math. (Berlin), Springer-Verlag, Berlin 1998.

[19] U. Bruzzo and R. Cianci, Differential equations, Frobenius theorem and local flows on supermanifolds. *J. Phys. A Math. Gen.* **18** (1985), 417–423. 111

[20] V. Chari and A. Pressley, *A guide to quantum groups*. Cambridge University Press, Cambridge 1994. 26, 113

[21] S.-J. Cheng and W. Wang, Dualities for Lie superalgebras. Preprint 2009. arXiv:1001.0074v2 [math.RT] 255

[22] P. Deligne and J. W. Morgan, Notes on supersymmetry (following Joseph Bernstein). In *Quantum fields and strings; a course for mathematicians*, Vol. 1, Amer. Math. Soc., Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999, 41–97. viii, 24, 26, 27, 53, 89, 111, 140, 153, 170, 229, 230

[23] M. Demazure and P. Gabriel, *Groupes algébriques*. Tome I: Géométrie algébrique, généralités, groupes commutatifs, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. 28, 41, 42, 43, 44, 183, 186, 188, 197, 201, 202, 210, 216, 262, 263

[24] M. Demazure and A. Grothendieck, *Schémas en groupes. I, II, III*. Séminaire de géométrie algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Math. 151, 152, 153, Springer-Verlag, Berlin 1970.

[25] B. DeWitt, *Supermanifolds*. Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge 1984. viii

[26] J. Dieudonné, *Éléments d'analyse*. Tome III: Chapitres XVI et XVII, Cahiers Scientifiques XXXIII, Gauthier-Villars Éditeur, Paris 1970. 82, 119, 273, 274

[27] J. Dixmier, *Enveloping algebras*. Grad. Stud. Math. 11, Amer. Math. Soc., Providence, RI, 1996. 249, 250

[28] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*. Grad. Texts in Math. 150, Springer-Verlag, New York 1995. 268

[29] D. Eisenbud and J. Harris, *The geometry of schemes*. Grad. Texts in Math. 197, Springer Verlag, New York 2000. 28, 33, 35, 36, 37, 39, 41, 43, 44, 65, 180, 183, 262

[30] R. Fioresi, On algebraic super-groups and quantum deformations. *J. Algebra Appl.* **2** (2003), 403–423. 206

[31] R. Fioresi and F. Gavarini, Chevalley supergroups. *Mem. Amer. Math. Soc.*, to appear; see also arXiv:0808.0785v5 [math.RA]. 140

[32] R. Fioresi, M. A. Lledó, and V. S. Varadarajan, The Minkowski and conformal superspaces. *J. Math. Phys.* **48** (2007), no. 11, 113505-1–27. 170, 172

[33] L. Frappat, A. Sciarrino, and P. Sorba, *Dictionary on Lie algebras and superalgebras*. Academic Press, Inc., San Diego, CA, 2000. 240

[34] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, Progress toward a theory of supergravity. *Phys. Rev. D* **13** (1976), 3214–3218. 140

[35] R. Giachetti and R. Ricci, **R**-actions, derivations, and Frobenius theorem on graded manifolds. *Adv. Math.* **62** (1986), 84–100. 111

[36] Yu. A. Gol'fand and E. P. Likhtman, Extension of the algebra of Poincaré group generators and violation of P invariance. *JETP Lett.* **13** (1971), 323–326. 140

[37] P. Griffiths and J. Harris, *Principles of algebraic geometry*. Wiley Classics Lib., John Wiley & Sons, Inc., New York 1994. 35, 44

[38] A. Grothendieck, *Fondaments de la géometrie algébrique*. Secretariat Mathematique, Paris 1962. 44, 263, 268

[39] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. In *Séminaire Bourbaki*, Vol. 2, Exp. No. 69, Soc. Math. France, Paris 1995, 193–200. 75

[40] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques. Reprint of *Bol. Soc. Mat. São Paulo* **8** (1953), 1–79; *Resenhas* **2** (1996), 401–480. 75

[41] A. Grothendieck, Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki*, Vol. 5, Exp. No. 190, Soc. Math. France, Paris 1995, 299–327. 263

[42] C. Gruson and V. Serganova, Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras. *Proc. London Math. Soc.* (3) **101** (2010), 852–892. 255

[43] R. Hartshorne, *Algebraic geometry*. Grad. Texts In Math. 52, Springer-Verlag, New York 1977. 28, 30, 33, 35, 36, 37, 43, 44, 176, 193, 265, 268

[44] V. G. Kac, Lie superalgebras. *Adv. Math.* **26** (1977), 8–26. 230, 231, 235, 236, 240, 241, 244, 250, 254

[45] V. G. Kac, Representations of classical Lie superalgebras. In *Differential geometrical methods in mathematical physics, II*, Lecture Notes in Math. 676, Springer-Verlag, Berlin 1978, 597–626. 231, 240, 250, 255

[46] V. G. Kac, A sketch of Lie superalgebras theory. *Comm. Math. Phys.* **53** (1977), 31–64.

[47] A. W. Knapp, *Advanced real analysis.* Cornerstones, Birkhäuser, Boston, Mass., 2005. 102

[48] I. Kolář, P. W. Michor, and J. Slovák, *Natural operations in differential geometry*. Springer-Verlag, Berlin 1993. 156

[49] B. Kostant, Graded manifolds, graded Lie theory, and prequantization. In *Differential geometrical methods in mathematical physics*, Lecture Notes in Math. 570, Springer-Verlag, Berlin 1977, 177–306. viii, 27, 53, 89, 112, 124, 140

[50] J.-L. Koszul, Graded manifolds and graded Lie algebras. In *Proceedings of the international meeting on geometry and physics*, Pitagora Editrice, Bologna 1983,71–84. 111, 112, 137, 140

[51] S. Lang, *Algebra*. Addison Wesley, Reading, Mass., 1984. 4, 44, 259, 265, 268

[52] S. Lang, *Real and functional analysis*. Grad. Texts in Math. 142, Springer-Verlag, New York 1993. 90

[53] D. A. Leites, Introduction to the theory of supermanifolds. *Russian Math. Surveys* **35** (1980), 1–64. viii, 89, 90, 102

[54] D. A. Leites, Quantization and supermanifolds. Appendix 3 in F. Berezin, M. Shubin *The Schrödinger equation*, Math. Appl. (Soviet Ser.) 66, Kluwer Academic Publishers Group, Dordrecht 1991. 97

[55] S. Maclane, *Categories for the working mathematician*, Grad. Texts in Math. 5, Springer-Verlag, New York 1998. 268

[56] Y. I. Manin, *Gauge field theory and complex geometry*. Translated by N. Koblitz and J. R. King, Grundlehren Math. Wiss. 289, Springer-Verlag, Berlin 1997. 27, 53, 89, 90, 102, 111, 172, 194, 201

[57] H. Matsumura, *Commutative ring theory*. Cambridge Stud. Adv. Math. 8, Cambridge University Press, Cambridge 1989. 268

[58] P. W. Michor, *Topics in differential geometry*. Grad. Stud. Math. 93, Amer. Math. Soc., Providence, RI, 2008. 76, 77, 84, 89

[59] S. Montgomery, *Hopf algebras and their actions on rings*. CBMS Regional Conf. Ser. in Math. 82 Amer. Math. Soc., Providence, RI, 1993. 26

[60] D. Mumford, *The red book of varieties and schemes*. Lecture Notes in Math. 1358, Springer-Verlag, Berlin 1988.

[61] J. Nestruev, *Smooth manifolds and observables*. Grad. Texts in Math. 220, Springer-Verlag, New York 2003.

[62] I. Penkov and V. Serganova, Generic irreducible representations of finite-dimensional Lie superalgebras. *Internat. J. Math.* **5** (1994), 389–419. 245, 251, 252

[63] I. Penkov, Characters of typical irreducible finite-dimensional $q(n)$-modules. *Funct. Anal. Appl.* **20** (1986), 30–37. 248

[64] W. Rudin, *Functional analysis*. Internat. Ser. Pure Appl. Math., McGraw Hill, New York 1991. 84, 269, 270

[65] A. Salam and J. Strathdee, Super-gauge transformations. *Nucl. Phys. B* **76** (1974), 477–482. vii, 140

[66] M. Scheunert, *The theory of Lie superalgebras*. Lecture Notes in Math. 716, Springer-Verlag, Berlin 1978.

[67] V. Serganova, Kac-Moody superalgebras and integrability. Preprint. 252, 254, 255

[68] V. Serganova, Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra $gl(m|n)$. *Selecta Math. (N.S.)* **2** (1996), 607–651. 255

[69] V. Serganova, On generalizations of root systems. *Comm. Algebra* **24** (1996), 4281–4299.

[70] J.-P. Serre, Faiseaux algebrique coherent. *Ann. of Math.* **61** (1955), 197–278. 44

[71] R. Steinberg, *Lectures on Chevalley groups*. Notes prepared by John Faulkner and Robert Wilson, Yale University, New Haven, Conn., 1968. 249

[72] M. E. Sweedler, *Hopf algebras*. Math. Lecture Note Ser., W. A. Benjamin, Inc., New York 1969. 26, 83, 120, 123

[73] F. Trèves, *Topological vector spaces, distributions and kernels*. Academic Press, New York 1967. 75, 273

[74] G. M. Tuynman, *Supermanifolds and supergroups*. Math. Appl. 570, Kluwer Academic Publishers, Dordrecht 2004.

[75] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*. Grad. Texts in Math. 102, Springer-Verlag, New York 1984. 125, 154, 157, 158, 245

[76] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction*. Courant Lect. Notes Math. 11, New York University, Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI, 2004. viii, 24, 27, 89, 90, 102, 111, 172, 268

[77] A. Vistoli, Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, Math. Surveys Monogr. 123, Amer. Math. Soc., Providence, RI, 2005, 1–104. 165, 263, 264, 268

[78] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*. Grad. Texts in Math. 94, Springer-Verlag, New York 1983. 61, 98, 102

[79] W. C. Waterhouse, *Introduction to affine group schemes.* Grad. Texts in Math. 66, Springer-Verlag, New York 1979.

[80] J. Wess and B. Zumino, Supergauge transformations in four dimensions. *Nucl. Phys. B* **70** (1974), 39–50. vii, 140

Index